第三讲：应力应变分析

弹塑性力学研究生核心课程

任晓丹

同济大学建筑工程系
www.renxiaodan.com

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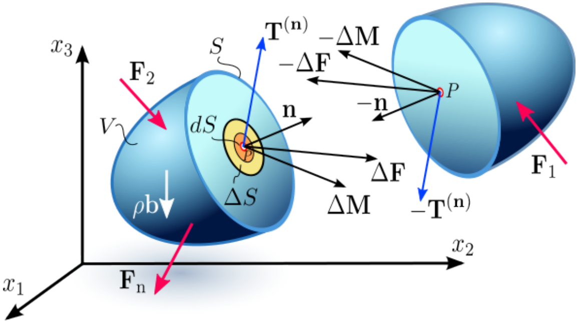
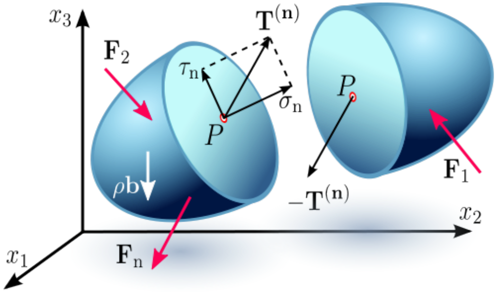
Euler–Cauchy 应力原理



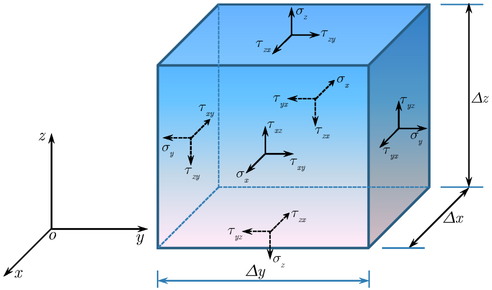
$$T_i^{(n)} = \lim_{\Delta s \to 0} \frac{\Delta F_i}{\Delta S} = \frac{\mathrm{d}F_i}{\mathrm{d}S} \qquad \lim_{\Delta s \to 0} \frac{\Delta M_i}{\Delta S} = 0$$

Euler–Cauchy 应力原理



$$\sigma_n = \lim_{\Delta s \to 0} \frac{\Delta F_n}{\Delta S} = \frac{\mathrm{d}F_n}{\mathrm{d}S} \qquad \tau = \lim_{\Delta s \to 0} \frac{\Delta F_s}{\Delta S} = \frac{\mathrm{d}F_s}{\mathrm{d}S}$$

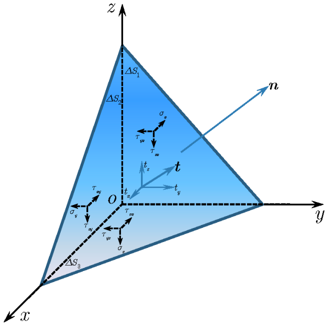
笛卡尔坐标系内的应力



$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

由于剪应力的互等性，上述应力张量中的 9 个分量只有 6 个是彼此独立的，即：应力张量是二阶对称张量。

斜截面上的应力



$$\begin{cases} t_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ t_y = \tau_{yx} n_x + \sigma_y n_y + \tau_{yz} n_z \\ t_z = \tau_{zx} n_x + \tau_{zy} n_y + \sigma_z n_z \end{cases}$$

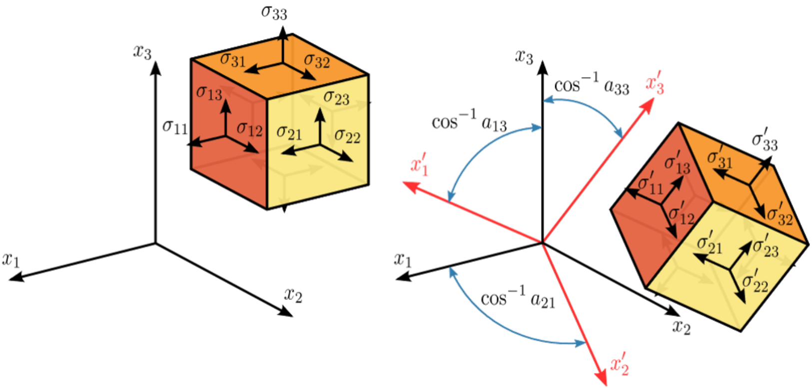
写成张量形式（Cauchy 应力公式）

$$t_i = \sigma_{ij} n_j, \quad \boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$$

斜截面上的正应力和剪应力

$$\begin{cases} \sigma_n = \boldsymbol{t} \cdot \boldsymbol{n} = n_i \sigma_{ij} n_j \\ \tau = \sqrt{\boldsymbol{t} \cdot \boldsymbol{t} - \sigma_n^2} \end{cases}$$

应力转轴


σ_{33}
x_3
σ_{31}
σ_{32}
σ_{13}
σ_{23}
σ_{11}
σ_{12}
σ_{21}
σ_{22}
x_1
x_2
x_3
$\cos^{-1} a_{33}$
x'_3
$\cos^{-1} a_{13}$
x'_1
σ'_{33}
σ'_{31}
σ'_{13}
σ'_{11}
σ'_{12}
σ'_{32}
σ'_{21}
σ'_{23}
σ'_{22}
x_1
$\cos^{-1} a_{21}$
x_2
x'_2

应力转轴公式

- 不同坐标系下 Cauchy 应力公式

$$t_i = \sigma_{ij} n_j \qquad t'_i = \sigma'_{ij} n'_j$$

- 向量坐标转换公式

$$n'_j = n_l \beta_{lj} \qquad t'_i = t_k \beta_{ki}$$

$$\beta_{ki} t_k = \sigma'_{ij} n_l \beta_{lj} \quad \Rightarrow \quad \beta_{mi} \beta_{ki} t_k = \delta_{mk} t_k = t_m = \beta_{mi} \sigma'_{ij} n_l \beta_{lj}$$

- 应力转轴公式 (张量表达)

$$\sigma_{ij} = \beta_{ik} \sigma'_{kl} \beta_{lj} \quad \Rightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\beta}^{\mathrm{T}} \cdot \boldsymbol{\sigma}' \cdot \boldsymbol{\beta}$$

应力转轴

$$
\begin{aligned}
\sigma'_{11} &= a_{11}^2\sigma_{11} + a_{12}^2\sigma_{22} + a_{13}^2\sigma_{33} + 2a_{11}a_{12}\sigma_{12} + 2a_{11}a_{13}\sigma_{13} + 2a_{12}a_{13}\sigma_{23},\\
\sigma'_{22} &= a_{21}^2\sigma_{11} + a_{22}^2\sigma_{22} + a_{23}^2\sigma_{33} + 2a_{21}a_{22}\sigma_{12} + 2a_{21}a_{23}\sigma_{13} + 2a_{22}a_{23}\sigma_{23},\\
\sigma'_{33} &= a_{31}^2\sigma_{11} + a_{32}^2\sigma_{22} + a_{33}^2\sigma_{33} + 2a_{31}a_{32}\sigma_{12} + 2a_{31}a_{33}\sigma_{13} + 2a_{32}a_{33}\sigma_{23},\\
\sigma'_{12} &= a_{11}a_{21}\sigma_{11} + a_{12}a_{22}\sigma_{22} + a_{13}a_{23}\sigma_{33}\\
&\quad + (a_{11}a_{22} + a_{12}a_{21})\sigma_{12} + (a_{12}a_{23} + a_{13}a_{22})\sigma_{23} + (a_{11}a_{23} + a_{13}a_{21})\sigma_{13},\\
\sigma'_{23} &= a_{21}a_{31}\sigma_{11} + a_{22}a_{32}\sigma_{22} + a_{23}a_{33}\sigma_{33}\\
&\quad + (a_{21}a_{32} + a_{22}a_{31})\sigma_{12} + (a_{22}a_{33} + a_{23}a_{32})\sigma_{23} + (a_{21}a_{33} + a_{23}a_{31})\sigma_{13},\\
\sigma'_{13} &= a_{11}a_{31}\sigma_{11} + a_{12}a_{32}\sigma_{22} + a_{13}a_{33}\sigma_{33}\\
&\quad + (a_{11}a_{32} + a_{12}a_{31})\sigma_{12} + (a_{12}a_{33} + a_{13}a_{32})\sigma_{23} + (a_{11}a_{33} + a_{13}a_{31})\sigma_{13}.
\end{aligned}
$$

Mohr 圆公式

主应力

- 存在某斜截面，只有正应力分量而没有剪应力分量，此时正应力方向为截面法向，于是有 $\boldsymbol{t} = \sigma \boldsymbol{n}$。
- 结合斜截面应力公式 $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$，得到应力特征方程

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \sigma \boldsymbol{n} \quad \Rightarrow \quad \left|\sigma_{ij} - \sigma\delta_{ij}\right| = 0$$

- 展开特征方程

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

- 特征方程有 3 个实根，即 3 个主应力 σ_1, σ_2 和 σ_3。一般约定：$\sigma_1 \geq \sigma_2 \geq \sigma_3$。进而可求得三个主方向，可以证明：当 $\sigma_1 \neq \sigma_2 \neq \sigma_3$ 时，三个主应力方向相互垂直。特征方程的系数，即应力张量三个不变量的求法，请参照上节课的讨论以及教材或其它资料。

应力偏量

- 定义静水应力：

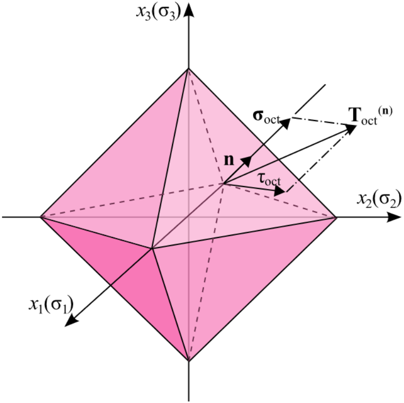
$$\sigma_m = \frac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

- 应力偏量定义为：

$$s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$$

- 应力偏量表示纯剪应力状态，对于很多材料，是其重要的破坏控制机制，所以应力偏量应用十分广泛。
- J_1、J_2 和 J_3 分别称为应力偏张量的第一、第二、第三不变量。由于 $J_1 = 0$，因此，一点的应力状态也可以用 I_1、J_2 和 J_3 表示。

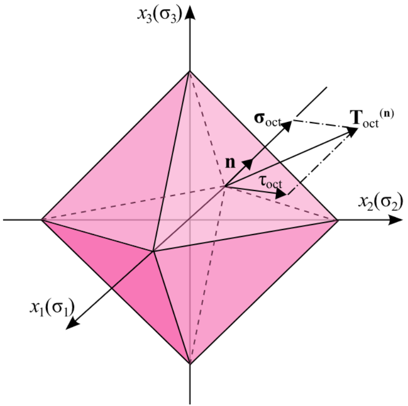
八面体应力



$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{T}_{\text{oct}}^{(\mathbf{n})} &= \sigma_{ij} n_i \mathbf{e}_j \\ &= \sigma_1 n_1 \mathbf{e}_1 + \sigma_2 n_2 \mathbf{e}_2 + \sigma_3 n_3 \mathbf{e}_3 \\ &= \frac{1}{\sqrt{3}} (\sigma_1 \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 + \sigma_3 \mathbf{e}_3) \end{aligned}$$

八面体应力



$$
\begin{aligned}
\sigma_{\text{oct}} &= T_i^{(n)} n_i = \sigma_{ij} n_i n_j \\
&= \sigma_1 n_1 n_1 + \sigma_2 n_2 n_2 + \sigma_3 n_3 n_3 \\
&= \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \tfrac{1}{3} I_1
\end{aligned}
$$

$$
\begin{aligned}
\tau_{\text{oct}} &= \sqrt{T_i^{(n)} T_i^{(n)} - \sigma_{\text{n}}^2} \\
&= \left[\frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right. \\
&\quad \left. - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2 \right]^{1/2} \\
&= \tfrac{1}{3}\sqrt{2 I_1^2 - 6 I_2} = \sqrt{\tfrac{2}{3} J_2}
\end{aligned}
$$

三维主应力空间

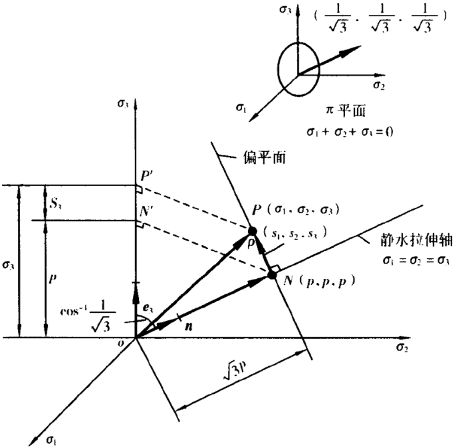
$$|ON| = OP\cdot n = (\sigma_1,\sigma_2,\sigma_3)\cdot\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

$$|ON| = \frac{1}{\sqrt{3}}\ (\sigma_1+\sigma_2+\sigma_3)\ = \frac{I_1}{\sqrt{3}} = \sqrt{3}p$$

$$NP = OP - ON$$

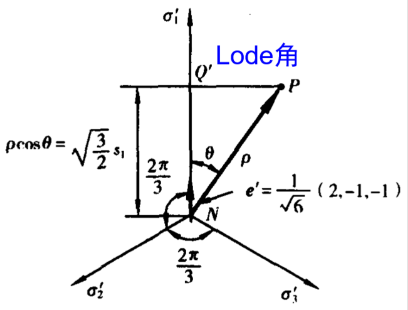
$$\begin{aligned} NP &= (\sigma_1,\ \sigma_2,\ \sigma_3) - (p,\ p,\ p) \\ &= [(\sigma_1-p),\ (\sigma_2-p),\ (\sigma_3-p)] \\ &= (s_1,\ s_2,\ s_3) \end{aligned}$$

$$\rho = |NP| = (s_1^2+s_2^2+s_3^2)^{1/2} = \sqrt{2J_2} = \sqrt{3}\tau_{oct}$$



Haigh-Westergaard stress space

Ⅱ 平面



$$NQ' = \rho\cos\theta = \boldsymbol{NP}\cdot\boldsymbol{e}'$$

$$= (s_1,\ s_2,\ s_3)\cdot\frac{1}{\sqrt{6}}(2,\ -1,\ -1)$$

$$= \frac{1}{\sqrt{6}}(2s_1 - s_2 - s_3) = \sqrt{\frac{3}{2}}s_1$$

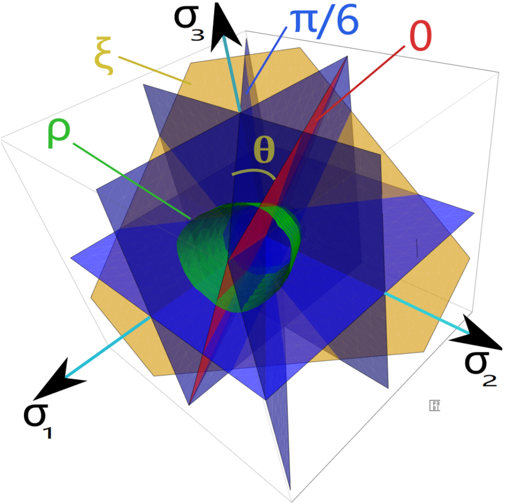
$$\cos\theta = \frac{\sqrt{3}}{2}\frac{s_1}{\sqrt{J_2}}$$

$$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$$

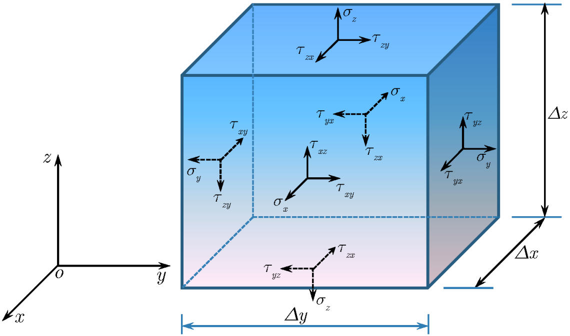
⇨

$$0 \leqslant \theta \leqslant \frac{\pi}{3}$$

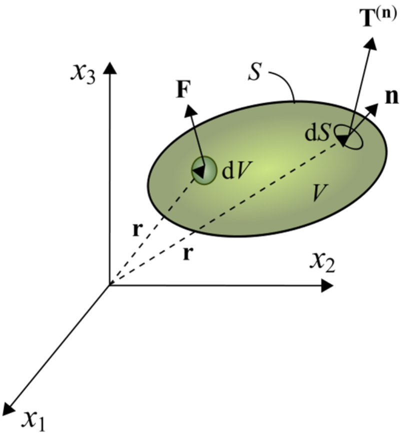
Lode/Haigh-Westergaard 坐标系



平衡方程


z
o
y
x
σ_z
τ_{zy}
τ_{zx}
σ_x
τ_{yx}
τ_{zx}
τ_{xy}
σ_y
τ_{zy}
τ_{xz}
τ_{xy}
σ_x
τ_{yz}
σ_y
τ_{yx}
Δz
Δx
τ_{zx}
τ_{yz}
σ_z
Δy

（第一）平衡方程



$$\oint_S \boldsymbol{T}^{(n)}\,\mathrm{d}S + \int_V \boldsymbol{F}\,\mathrm{d}V = 0$$

因为 $\boldsymbol{T}^{(n)} = \boldsymbol{\sigma}\cdot\boldsymbol{n}$，并使用散度定理，有

$$\int_V (\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{F})\,\mathrm{d}V = 0$$

上式对任意体积 V 均成立，有

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{F} = 0$$

（第二）平衡方程

- 合力矩为零

$$\boldsymbol{M}_O = \int_{\partial\Omega} \boldsymbol{r} \times \boldsymbol{T} \mathrm{d}S + \int_{\Omega} \boldsymbol{r} \times \boldsymbol{F} \mathrm{d}\Omega$$
$$= \int_{\partial\Omega} \epsilon_{ijk} x_j T_k^n \mathrm{d}S + \int_{\Omega} \epsilon_{ijk} x_j F_k \mathrm{d}\Omega = 0$$

- 第一项

$$R_1 = \int_{\partial\Omega} \epsilon_{ijk} x_j \sigma_{mk} n_m \mathrm{d}S = \int_{\Omega} (\epsilon_{ijk} x_j \sigma_{mk})_{,m} \mathrm{d}\Omega$$

$$= \int_{\Omega} (\epsilon_{ijk} x_{j,m} \sigma_{mk} + \epsilon_{ijk} x_j \sigma_{mk,m}) \mathrm{d}\Omega$$

（第二）平衡方程

- 回代

$$M_O = \int_\Omega (\epsilon_{ijk} x_{j,m} \sigma_{mk})\, d\Omega + \int_\Omega \epsilon_{ijk} x_j (\sigma_{mk,m} + F_k)\, d\Omega = 0$$

- 由第一平衡方程，可得上式第二项等于零，第一项有

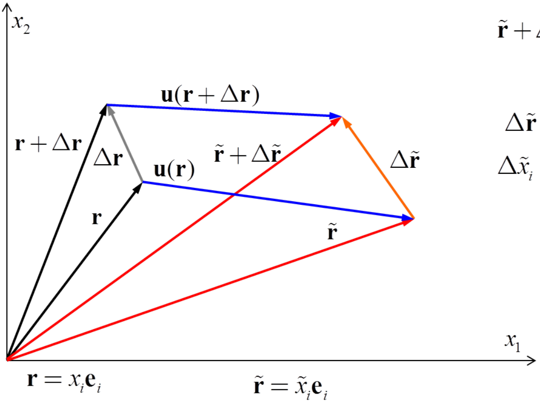
$$\int_\Omega (\epsilon_{ijk} x_{j,m} \sigma_{mk})\, d\Omega = \int_\Omega (\epsilon_{ijk} \delta_{jm} \sigma_{mk})\, d\Omega = \int_\Omega (\epsilon_{ijk} \sigma_{jk})\, d\Omega = 0$$

- 上式对于任意体积成立

$$\epsilon_{ijk} \sigma_{jk} = 0 \quad \Rightarrow \quad \sigma_{ij} = \sigma_{ji}$$

剪应力互等！

变形分析



$$\tilde{\mathbf{r}}+\Delta\tilde{\mathbf{r}}=\mathbf{r}+\Delta\mathbf{r}+\mathbf{u}(\mathbf{r}+\Delta\mathbf{r})$$

$$\Downarrow$$

$$\Delta\tilde{\mathbf{r}}=\Delta\mathbf{r}+\mathbf{u}(\mathbf{r}+\Delta\mathbf{r})-\mathbf{u}(\mathbf{r})$$

$$\Delta\tilde{x}_i=\Delta x_i+u_i(\mathbf{r}+\Delta\mathbf{r})-u_i(\mathbf{r})$$

$$\Downarrow$$

$$\Delta\tilde{x}_i\approx\Delta x_i+u_{i,j}\Delta x_j$$
$$=(\delta_{ij}+u_{i,j})\Delta x_j$$

格林应变

考虑微段的长度变化

$$
\begin{aligned}
||\mathrm{d}\tilde{\boldsymbol{r}}||^2 = \mathrm{d}\tilde{\boldsymbol{r}}\cdot\mathrm{d}\tilde{\boldsymbol{r}} = \mathrm{d}\tilde{x}_i\mathrm{d}\tilde{x}_i &= \mathrm{d}x_j(\delta_{ij}+u_{i,j})(\delta_{ik}+u_{i,k})\mathrm{d}x_k \\
&= \mathrm{d}x_j(\delta_{jk}+u_{j,k}+u_{k,j}+u_{i,j}u_{i,k})\mathrm{d}x_k \\
&= \mathrm{d}\boldsymbol{r}\cdot(\mathbf{1}+2\boldsymbol{E})\cdot\mathrm{d}\boldsymbol{r}
\end{aligned}
$$

此处格林（Green）应变定义为

$$
E_{ik} = \frac{1}{2}(u_{j,k}+u_{k,j}+u_{i,j}u_{i,k})
$$

定义变形梯度

$$
\boldsymbol{F} = \nabla\otimes\boldsymbol{u} \qquad F_{ij} = u_{i,j}
$$

格林（Green）应变

$$
\boldsymbol{E} = \boldsymbol{F}+\boldsymbol{F}^{\mathrm{T}}+\boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{F}
$$

格林应变与线性应变

定义位移梯度的对称与反对称分量

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

最终有如下表达式

$$E_{ij} = \varepsilon_{ij} + \frac{1}{2}(\varepsilon_{ik}\varepsilon_{jk} - \varepsilon_{ik}\omega_{jk} - \omega_{ik}\varepsilon_{jk} + \omega_{ik}\omega_{jk})$$

应变的分解与不变量

- 应变的球偏分解
- 体积应变
- 应变的不变量
- 与应力张量分析类似，均可利用二阶张量的相关结果，不再重复讲解

协调条件

u_i ⇨ $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

ε_{ij} plus Compatibility ⇨ u_i

⇩

$$\varepsilon_{ik,jl} - \varepsilon_{jk,il} - \varepsilon_{il,jk} + \varepsilon_{jl,ik} = 0$$

弹性本构关系

In physics, elasticity is the ability of a body to resist a distorting influence or stress and to return to its original size and shape when the stress is removed.

- Linear elasticity
- Finite elaticity
  - Cauchy elasticity (柯西弹性)
  - Hyperelasticity or Green elasticity (超弹性、格林弹性)
  - Hypoelasticity (次弹性)

线弹性本构关系

- 线弹性本构关系

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \ , \ \boldsymbol{\sigma} = \mathbb{E}_0 : \boldsymbol{\varepsilon}$$

- 刚度张量的对称性
  - 次对称性

$$E_{ijkl} = E_{jikl} = E_{ijlk}$$

来源于：

$$\sigma_{ij} = \sigma_{ji} \qquad \varepsilon_{kl} = \varepsilon_{lk}$$

  - 主对称性

$$E_{ijkl} = E_{klij}$$

来源于：

$$W = \frac{1}{2} E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

线弹性本构关系

- Voight notation

s_α	α	1	2	3	4	5	6
s_{ij}	i	1	2	3	1	1	2
	j	1	2	3	2	3	3

- 刚度张量矩阵化

$$[E] = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} \\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46} \\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} \\ E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66} \end{bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1112} & E_{1113} & E_{1123} \\ E_{2211} & E_{2222} & E_{2233} & E_{2212} & E_{2213} & E_{2223} \\ E_{3311} & E_{3322} & E_{3333} & E_{3312} & E_{3313} & E_{3323} \\ E_{1211} & E_{1222} & E_{1213} & E_{1212} & E_{1213} & E_{1223} \\ E_{1311} & E_{1322} & E_{1333} & E_{1312} & E_{1313} & E_{1323} \\ E_{2311} & E_{2322} & E_{2333} & E_{2312} & E_{2313} & E_{2323} \end{bmatrix}$$

线弹性本构关系

正交各向异性 (Orthotropy)

$$[E] = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & 0 \\ E_{21} & E_{22} & E_{23} & 0 & 0 & 0 \\ E_{31} & E_{32} & E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{66} \end{bmatrix}$$

线弹性本构关系

各向同性 (Isotropy)

$$E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{kl} + \delta_{il}\delta_{jk})$$

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

$$\sigma_{kk} = 2K\varepsilon_{kk} \ , \ \ s_{ij} = 2\mu e_{ij}$$

$$W = \frac{1}{2}K(\varepsilon_{kk})^2 + \mu e_{ij}e_{ij}$$

柯西弹性

一般的，柯西弹性本构关系是指柯西应力 $\boldsymbol{\sigma}$ 与变形梯度 $\boldsymbol{F}$ 之间满足如下函数关系：

$$\boldsymbol{\sigma} = \mathcal{G}(\boldsymbol{F})$$

对于工程中常见的情况，用小（对称）应变代替变形梯度，有

$$\boldsymbol{\sigma} = \mathcal{G}(\boldsymbol{\varepsilon})$$

上式表明，当前的应力只与当前的应变（变形）有关，与加载历史无关，所描述的加载过程完全可逆。上述式子定义的材料称为柯西弹性材料。

柯西弹性

应变能函数

$$W(\varepsilon_{ij}) = \int_0^{\varepsilon_{ij}} \sigma_{ij} \mathrm{d}\varepsilon_{ij} = \int_0^{\boldsymbol{\varepsilon}} \mathcal{G}(\boldsymbol{\varepsilon}) : \mathrm{d}\boldsymbol{\varepsilon}$$

应变能变化

$$W(\varepsilon_{ij}^*, \varepsilon_{ij}) = \int_{\varepsilon_{ij}^*}^{\varepsilon_{ij}} \sigma_{ij} \mathrm{d}\varepsilon_{ij} = \int_{\boldsymbol{\varepsilon}^*}^{\boldsymbol{\varepsilon}} \mathcal{G}(\boldsymbol{\varepsilon}) : \mathrm{d}\boldsymbol{\varepsilon}$$

环路上的应变能

$$W(\varepsilon_{ij}^*) = \oint_{\varepsilon_{ij}^*} \sigma_{ij} \mathrm{d}\varepsilon_{ij} = \oint_{\boldsymbol{\varepsilon}^*} \mathcal{G}(\boldsymbol{\varepsilon}) : \mathrm{d}\boldsymbol{\varepsilon}$$

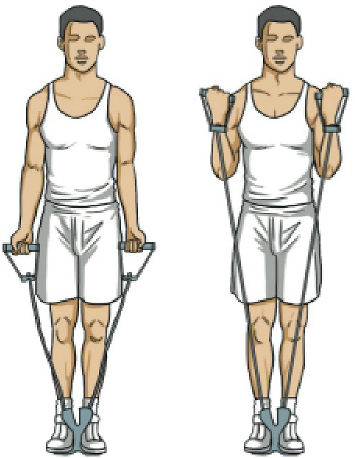
柯西弹性

环路上的应变能

$$W(\varepsilon_{ij}^*) = \oint_{\varepsilon_{ij}^*} \sigma_{ij} \mathrm{d}\varepsilon_{ij} = \oint_{\boldsymbol{\varepsilon}^*} \mathcal{G}(\boldsymbol{\varepsilon}) : \mathrm{d}\boldsymbol{\varepsilon} = 0?$$

- 对于柯西弹性材料，上式可能不满足。即对于特定的加载循环，可能出现材料净输出能量（spurious energy）的情况，将不符合热力学定律。表现在实际应用中，材料模型的分析、计算模拟等将可能出现各类不稳定现象。
- 为了满足上式的要求，基于应变能提出了超弹性（格林弹性）材料模型。

超弹性（格林弹性）



- 应变能函数 W 存在
- 应力与应变的关系由应变能的偏导数确定。
- 常用于橡胶等材料，一般是大变形问题。
- 存在是应力与应变的配对问题。

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{E}} \quad , \quad \boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{F}}$$

- 对于小变形问题: $\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}$。

超弹性（格林弹性）

- 一般将应变能 W 假定为应变量度（ε、$\boldsymbol{F}$、$\boldsymbol{E}$ 等）不变量的函数，有

$$W = W(I_1', I_2', I_3') = W(\lambda_1', \lambda_2', \lambda_3')$$

- 求导可得对应应力

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{E}} \quad , \quad \boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{F}} \quad , \quad \boldsymbol{\sigma} = \frac{\partial W}{\partial \varepsilon}$$

超弹性（格林弹性）

- Neo-Hookean 模型

$$W = C_1(I_1 - 3)\ ,\quad I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2\ ,\quad \text{不可压缩模型}$$

$$W = C_1\ (\bar{I}_1 - 3) + D_1\ (J - 1)^2\ ,\quad \text{可压缩模型}$$

$$J = \det(\boldsymbol{F}) = \lambda_1\lambda_2\lambda_3\ ,\quad \bar{I}_1 = J^{-1} I_1$$

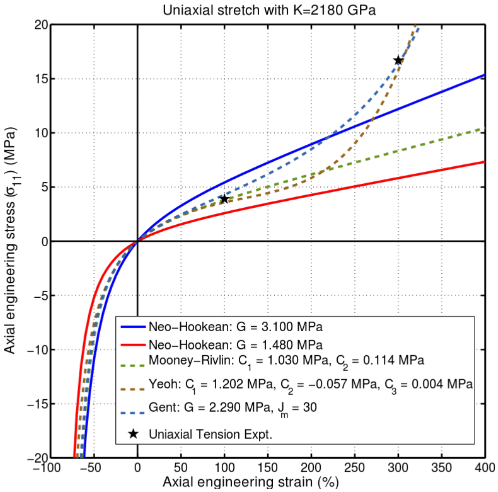
- Mooney–Rivlin 模型

$$W = C_1(\bar{I}_1 - 3) + C_2(\bar{I}_2 - 3)$$

$$\bar{I}_1 := J^{-2/3}\ I_1\ ;\quad I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2\ ;\quad J = \det(\boldsymbol{F}) = \lambda_1\lambda_2\lambda_3$$

$$\bar{I}_2 := J^{-4/3}\ I_2\ ;\quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$

超弹性（格林弹性）


Uniaxial stretch with K=2180 GPa
Axial engineering stress (σ11) (MPa)
Axial engineering strain (%)
20
15
10
5
0
−5
−10
−15
−20
−100
−50
0
50
100
150
200
250
300
350
400
Neo−Hookean: G = 3.100 MPa
Neo−Hookean: G = 1.480 MPa
Mooney−Rivlin: C1 = 1.030 MPa, C2 = 0.114 MPa
Yeoh: C1 = 1.202 MPa, C2 = −0.057 MPa, C3 = 0.004 MPa
Gent: G = 2.290 MPa, Jm = 30
Uniaxial Tension Expt.

次弹性

A hypoelastic material can be rigorously defined as one that is modeled using a constitutive equation satisfying the following two criteria (Truesdell 1963, 2004) :

- The Cauchy stress $\boldsymbol{\sigma}$ at time t depends only on the order in which the body has occupied its past configurations, but not on the time rate at which these past configurations were traversed. As a special case, this criterion includes a Cauchy elastic material, for which the current stress depends only on the current configuration rather than the history of past configurations.
- There is a tensor-valued function G such that $\dot{\boldsymbol{\sigma}} = G(\boldsymbol{\sigma}, \boldsymbol{L})$ is the material rate of the Cauchy stress tensor, and $\boldsymbol{L} = \dot{\boldsymbol{F}}$ is the spatial velocity gradient tensor.

次弹性

Specific formulations of hypoelastic models typically employ a so-called objective stress rate so that the G function exists only implicitly. Hypoelastic material models frequently take the form:

$$\overset{\circ}{\boldsymbol{\tau}} = \mathbb{M} : \dot{\boldsymbol{\varepsilon}}$$

- $\overset{\circ}{\boldsymbol{\tau}}$ is an objective rate of the Kirchhoff stress ($\boldsymbol{\tau} := J\boldsymbol{\sigma}$);
- $\mathbb{M}$ is the so-called elastic tangent stiffness tensor.

次弹性

- 客观应力率 $\overset{\circ}{\tau}$ 的定义和选取是连续介质力学的重要内容，囿于课程时间和内容限制，这里不再详细介绍。
- 对于小变形问题，有如下简化表达式：

$$\dot{\boldsymbol{\sigma}} = \mathbb{M} : \dot{\boldsymbol{\varepsilon}}$$

- 次弹性模型只是定义了增量形式，是一类非常一般性的模型，超弹性模型，弹塑性增量模型等均可以看做其特例。

谢谢！

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