

第十三、十四、十五讲 几何非线性问题初步

有限单元法研究生核心课程

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June 19, 2016

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概述

Total Lagrangian Formulation
Update Lagrangian Formulation
结构稳定问题简述

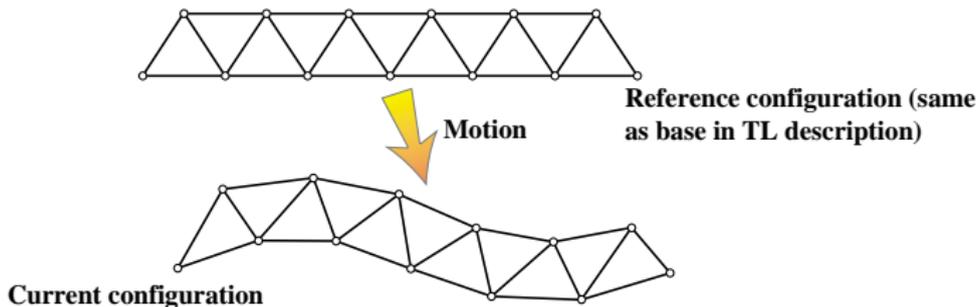
几何非线性问题



几何非线性问题

- 变形前后的位型差别很大，其几何度量发生很大变化。
- 应力、应变等状态量均与几何度量有关，此时建立的控制方程就与变形后的位型有关，成为隐式非线性方程。
- 根据问题性质的不同，几何非线性问题有“大应变”和“大位移”等说法，前者指问题的问题中位移梯度较大，后者指问题中的位移较大，特别是后者中，转动较大引起的问题尤为突出，此时称为“大转动”问题。
- 特别是在结构稳定（屈曲）分析中，几何非线性问题须合理考虑。

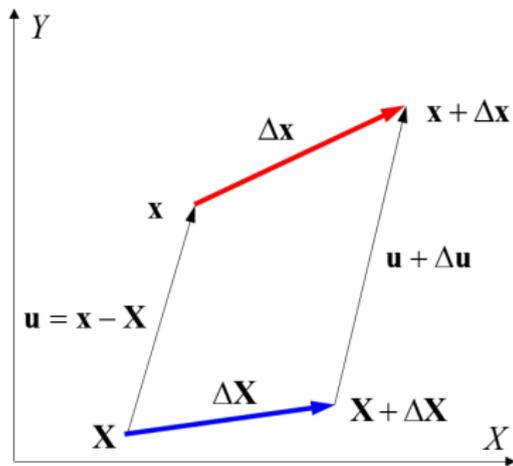
杆系大位移问题



Definition (Total Lagrangian Formulation)

将状态量映射到初始位型，在初始位型上建立控制方程的格式。

几何度量分析



- 一点的坐标，变形前为 X ，变形后为 x ，满足：

$$x = \phi(X, t), X = \varphi(x, t)$$

- 微元段 ΔX , Δx 满足：

$$\begin{cases} \Delta x = \frac{\partial x}{\partial X} \Delta X = F \Delta X \\ \Delta X = \frac{\partial X}{\partial x} \Delta x = F^{-1} \Delta x \end{cases}$$

$F = \frac{\partial x}{\partial X}$ 为变形矩阵，其行列式 $J = |F|$ 为 Jacobian 行列式。

几何度量分析

- 若不考虑长度变化

$$\|\Delta \boldsymbol{x}\|^2 = \Delta \boldsymbol{x}^T \Delta \boldsymbol{x} = \Delta \boldsymbol{X}^T \boldsymbol{F}^T \boldsymbol{F} \Delta \boldsymbol{X} = \Delta \boldsymbol{X}^T \boldsymbol{\Lambda} \Delta \boldsymbol{X}$$

可得 $\boldsymbol{F}^T \boldsymbol{F} = \boldsymbol{\Lambda}$ 。

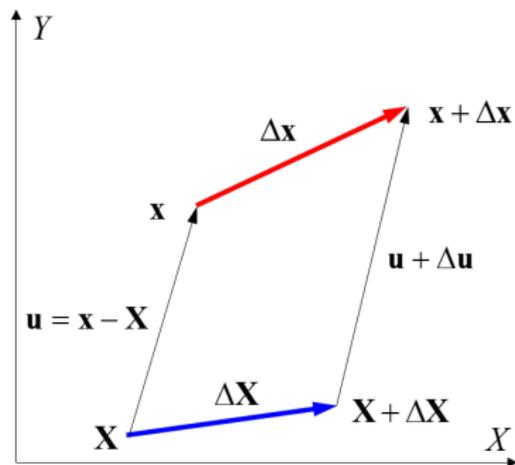
此时 \boldsymbol{F} 为正交矩阵，表示刚体转动，与坐标转换矩阵一致。

- 若考虑长度变化，有变形矩阵分解

$$\boldsymbol{F} = \boldsymbol{R} \boldsymbol{U}$$

\boldsymbol{R} 为正交矩阵，表示转动； \boldsymbol{U} 表示变形。

运动分析



- 位移满足:

$$\mathbf{u} = \mathbf{x} - \mathbf{X}$$

- 位移增量

$$\Delta \mathbf{u} = \Delta \mathbf{x} - \Delta \mathbf{X}$$

进一步有

$$\begin{aligned} \Delta \mathbf{x} &= \Delta \mathbf{X} + \Delta \mathbf{u} \\ &= \Delta \mathbf{X} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \Delta \mathbf{X} \\ &= (\mathbf{\Lambda} + \partial_X \mathbf{u}) \Delta \mathbf{X} \end{aligned}$$

运动分析

■ 长度分析

$$\begin{aligned} \|\Delta \mathbf{x}\|^2 &= \Delta x_I \Delta x_I = \Delta X_I (\delta_{KI} + u_{K,I}) (\delta_{KJ} + u_{K,J}) \Delta X_J \\ &= \Delta \mathbf{X} (\mathbf{1} + 2\mathbf{E}) \Delta \mathbf{X} \end{aligned}$$

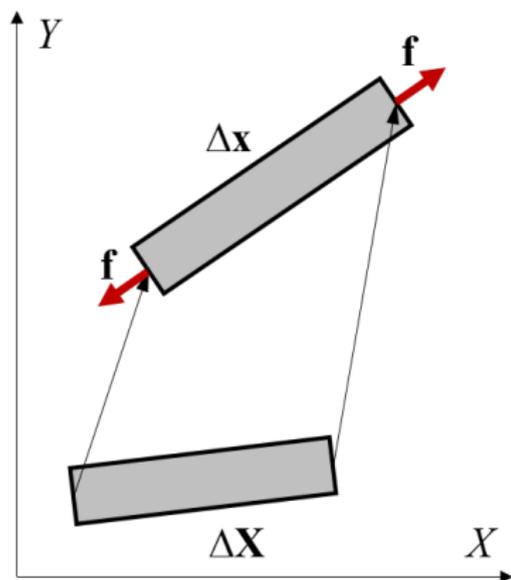
■ 有 Green 应变定义

$$E_{IJ} = \frac{1}{2} \left(\underbrace{u_{I,J} + u_{J,I}}_{\text{linear}} + \underbrace{u_{K,I} u_{K,J}}_{\text{quadratic}} \right)$$

■ 对于二维杆元

$$E = \frac{L^2 - L_0^2}{2L_0^2}$$

应力分析



- Cauchy(真实) 应力

$$\sigma = \frac{f}{A}$$

- 二维应力状态

$$\sigma \cdot nA = f = fn$$

可解的杆单元 Cauchy 应力

$$\sigma = \frac{f}{A} nn^T$$

A 和 n 均未知，很难列式。

应力分析

- 考虑几何度量采用初始位型，力采用当前状态

$$P \cdot n_0 A_0 = f = fn \Rightarrow P \cdot n_0 = \frac{f}{A_0} n$$

- 定义 P 为第一类 Piola-Kirchhoff 应力，几何非线性分析中常用，但是不具对称性。
- 与 Cauchy 的关系可以容易得到。

应力分析

- 考虑几何度量采用初始位型，同时将力映射回初始构型

$$\boldsymbol{S} \cdot \boldsymbol{n}_0 A_0 = \boldsymbol{f}_0 = \boldsymbol{F}^{-1} \boldsymbol{f} \Rightarrow \boldsymbol{S} \cdot \boldsymbol{n}_0 = \frac{f}{A_0} \boldsymbol{F}^{-1} \boldsymbol{n}$$

进一步化简可得

$$\boldsymbol{S} = \frac{f}{A_0} \boldsymbol{n}_0 \boldsymbol{n}_0^T$$

- 定义 \boldsymbol{S} 为第二类 Piola-Kirchhoff 应力，具有对称性，为理想的定义在初始构型上的应力。

平衡方程

- 对于变形后状态，杆元满足平衡方程

$$\frac{\partial \sigma}{\partial l} = 0$$

- 进一步得杆元虚功方程

$$\int_L A \sigma \delta \varepsilon dl = \mathbf{f}_{ext} \delta \mathbf{u}$$

- 所有状态变量均与当前位型有关，很难进一步分析求解。

平衡方程

- 将积分映射到初始构型

$$\sigma = S \frac{A_0}{A} \quad \delta \varepsilon = \delta \left(\frac{L - L_0}{L_0} \right) \quad dl = \frac{L}{L_0} dl_0$$

- 虚功原理

$$\int_L A \sigma \delta \varepsilon dl = \int_{L_0} A S \frac{A_0}{A} \frac{\delta L}{L_0} \frac{L}{L_0} dl_0 = \int_{L_0} A_0 S \delta E dl_0 = \mathbf{f}_{ext} \delta \mathbf{u}$$

其中

$$\delta E = \delta \left(\frac{1}{2} \frac{L^2 - L_0^2}{L_0^2} \right) = \frac{\delta L}{L_0} \frac{L}{L_0}$$

- 第二类 P-K 应力与 Green 应变能量对偶！
- 虚功方程为关于位移 \mathbf{u} 的非线性方程。

非线性方程的求解

- 非线性方程

$$\mathbf{F}(\mathbf{u}) = 0$$

- 关于增量做 Taylor 展开

$$\mathbf{F}(\mathbf{u}_0 + \Delta \mathbf{u}) = \mathbf{F}(\mathbf{u}_0) + \left. \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \right|_0 \Delta \mathbf{u} + \text{high order terms} = 0$$

增量求解

$$\Delta \mathbf{u} \approx -\mathbf{K}^{-1} \mathbf{F}(\mathbf{u}_0) \quad , \quad \mathbf{K} = \left. \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \right|_0 \text{ 为切线刚度矩阵}$$

- 迭代格式

$$\Delta \mathbf{u}_k = -\mathbf{K}_k^{-1} \mathbf{F}(\mathbf{u}_k) \quad , \quad \mathbf{u}_{k+1} = \mathbf{u}_k + \Delta \mathbf{u}_k$$

平衡方程摄动分析

■ 状态量增量

$${}^{t+\Delta t}\mathbf{u} = {}^t\mathbf{u} + {}^t\Delta\mathbf{u} \quad , \quad {}^{t+\Delta t}E = {}^tE + {}^t\Delta E \quad , \quad {}^{t+\Delta t}S = {}^tS + {}^t\Delta S$$

■ 增量型应力 - 应变关系 : ${}^t\Delta S = {}^tC_T {}^t\Delta E$

■ 内力虚功

$$\begin{aligned} \int_{L_0} A_0 ({}^{t+\Delta t}S) \delta ({}^{t+\Delta t}E) dl_0 &= \int_{L_0} A_0 ({}^tS + {}^tC_T {}^t\Delta E) \delta ({}^t\Delta E) dl_0 \\ &= \int_{L_0} A_0 ({}^t\Delta E) {}^tC_T \delta ({}^t\Delta E) dl_0 + \int_{L_0} A_0 ({}^tS) \delta ({}^t\Delta E) dl_0 \end{aligned}$$

■ 外力虚功

$${}^{t+\Delta t}\mathbf{f}_{ext} \delta ({}^{t+\Delta t}\mathbf{u}) = {}^{t+\Delta t}\mathbf{f}_{ext} \delta ({}^t\mathbf{u} + {}^t\Delta\mathbf{u}) = {}^{t+\Delta t}\mathbf{f}_{ext} \delta ({}^t\Delta\mathbf{u})$$

应变摄动分析

- 微元段上的 Green 应变

$$E = \frac{1}{2} \frac{\Delta L^2 - \Delta L_0^2}{\Delta L_0^2}$$

- 定义沿着杆轴线方向的几何增量：

$$\begin{cases} \Delta L^2 = \Delta x^2 + \Delta y^2 \\ \Delta L_0^2 = \Delta X^2 + \Delta Y^2 \end{cases} \quad \begin{cases} \Delta u_x = \Delta x - \Delta X \\ \Delta u_y = \Delta y - \Delta Y \end{cases} \quad \begin{cases} c_X = \frac{\Delta X}{\Delta L_0} = \frac{\partial X}{\partial L_0} \\ c_Y = \frac{\Delta Y}{\Delta L_0} = \frac{\partial Y}{\partial L_0} \end{cases}$$

- 代入可得

$$E = \underbrace{c_X \frac{\partial u_x}{\partial L_0} + c_Y \frac{\partial u_y}{\partial L_0}}_{\text{linear}} + \underbrace{\frac{1}{2} \left(\frac{\partial u_x}{\partial L_0} \right)^2 + \frac{1}{2} \left(\frac{\partial u_y}{\partial L_0} \right)^2}_{\text{quadratic}}$$

应变摄动分析

■ 应变增量

$$\Delta E = \underbrace{c_X \frac{\partial \Delta u_x}{\partial L_0} + c_Y \frac{\partial \Delta u_y}{\partial L_0}}_{\text{linear-stretching}} + \underbrace{\frac{\partial u_x}{\partial L_0} \frac{\partial \Delta u_x}{\partial L_0} + \frac{\partial u_y}{\partial L_0} \frac{\partial \Delta u_y}{\partial L_0}}_{\text{linear-rotation}} + \underbrace{\frac{1}{2} \left(\frac{\partial \Delta u_x}{\partial L_0} \right)^2 + \frac{1}{2} \left(\frac{\partial \Delta u_y}{\partial L_0} \right)^2}_{\text{quadratic}}$$

■ 定义应变增量的分解及其变分

$$\Delta E = \Delta e + \Delta \eta, \quad \delta \Delta E = \delta \Delta e + \delta \Delta \eta$$

其中 Δe 为线性应变增量， $\Delta \eta$ 为二次应变增量；线性应变变分项 $\delta \Delta e$ 与位移增量 Δu 无关，非线性应变变分项 $\delta \Delta \eta$ 为位移增量 Δu 的线性函数。

平衡方程摄动分析

■ 增量虚功原理第一项

$$\begin{aligned} \int_{L_0} A_0({}^t\Delta E)^t C_T \delta({}^t\Delta E) dl_0 &= \int_{L_0} A_0({}^t\Delta e + {}^t\Delta\eta)^t C_T (\delta^t\Delta e + \delta^t\Delta\eta) dl_0 \\ &= \underbrace{\int_{L_0} A_0({}^t\Delta e)^t C_T \delta({}^t\Delta e) dl_0}_{\text{linear}} + \text{high order terms} \end{aligned}$$

■ 增量虚功原理第二项

$$\int_{L_0} A_0({}^tS) \delta({}^t\Delta E) dl_0 = \underbrace{\int_{L_0} A_0({}^tS) \delta({}^t\Delta e) dl_0}_{\text{constant}} + \underbrace{\int_{L_0} A_0({}^tS) \delta({}^t\Delta\eta) dl_0}_{\text{linear}}$$

平衡方程摄动分析

- 增量虚功原理最终表达式

$$\int_{L_0} ({}^t\Delta e)^t C_T A_0 \delta({}^t\Delta e) dl_0 + \int_{L_0} A_0 ({}^tS) \delta({}^t\Delta \eta) dl_0$$

$$= {}^{t+\Delta t} \mathbf{f}_{ext} \delta({}^t\Delta \mathbf{u}) - \int_{L_0} A_0 ({}^tS) \delta({}^t\Delta e) dl_0$$

- 上式为位移增量 $\Delta \mathbf{u}$ 的线性积分方程，有限元插值后可化为线性代数方程组直接求解，进而代入牛顿法迭代过程求解非线性平衡方程在 $t + \Delta t$ 时刻的精确解。
- 左端第一项为材料刚度的贡献，第二项为初应力的贡献，右端第二项为内力。

二维杆两节点单元

- 位移向量表示为： $\mathbf{u} = [u_1, v_1, u_2, v_2]^T$
- 对于两节点杆单元，杆内应变为常数，有

$$\begin{aligned}
 E &= c_X \frac{\partial u_x}{\partial L_0} + c_Y \frac{\partial u_y}{\partial L_0} + \frac{1}{2} \left(\frac{\partial u_x}{\partial L_0} \right)^2 + \frac{1}{2} \left(\frac{\partial u_y}{\partial L_0} \right)^2 \\
 &= c_X \frac{u_2 - u_1}{L_0} + c_Y \frac{v_2 - v_1}{L_0} + \frac{1}{2} \left(\frac{u_2 - u_1}{L_0} \right)^2 + \frac{1}{2} \left(\frac{v_2 - v_1}{L_0} \right)^2 \\
 &= \mathbf{B}_0 \mathbf{u} + \frac{1}{2} \mathbf{u}^T \mathbf{H} \mathbf{u}
 \end{aligned}$$

$$\mathbf{B}_0 = \frac{1}{L_0} [-c_X, -c_Y, c_X, c_Y] \quad , \quad \mathbf{H} = \frac{1}{L_0^2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

二维两节点杆单元

■ 应变增量

$$\Delta E = (\mathbf{B}_0 + \mathbf{u}^T \mathbf{H}) \Delta \mathbf{u} + \frac{1}{2} \Delta \mathbf{u}^T \mathbf{H} \Delta \mathbf{u} = \mathbf{B} \Delta \mathbf{u} + \frac{1}{2} \Delta \mathbf{u}^T \mathbf{H} \Delta \mathbf{u}$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{u}^T \mathbf{H} = \frac{1}{L} [-c_x, -c_y, c_x, c_y]$$

$$\Delta e = \mathbf{B} \Delta \mathbf{u} = (\mathbf{B}_0 + \mathbf{u}^T \mathbf{H}) \Delta \mathbf{u} \quad \Delta \eta = \frac{1}{2} \Delta \mathbf{u}^T \mathbf{H} \Delta \mathbf{u}$$

■ 应变 (位移) 增量变分

$$\delta \Delta E = \delta \Delta e + \delta \Delta \eta$$

$$\delta \Delta e = \mathbf{B} \delta \Delta \mathbf{u} = (\mathbf{B}_0 + \mathbf{u}^T \mathbf{H}) \delta \Delta \mathbf{u} \quad \delta \Delta \eta = \Delta \mathbf{u}^T \mathbf{H} \delta \Delta \mathbf{u}$$

二维两节点杆单元

- 增量虚功原理化为代数方程

$$({}^t\mathbf{K}_M + {}^t\mathbf{K}_G) {}^t\Delta\mathbf{u} = {}^{t+\Delta t}\mathbf{f}_{ext} - {}^t\mathbf{f}_{int}$$

式中各项

$${}^t\mathbf{K}_M = {}^t\mathbf{B}^T ({}^t\mathbf{C}_T \mathbf{A}_0) {}^t\mathbf{B} \mathbf{L}_0 \quad \text{材料刚度矩阵}$$

$${}^t\mathbf{K}_G = \mathbf{A}_0 {}^t\mathbf{S} \mathbf{H} \mathbf{L}_0 \quad \text{几何刚度矩阵}$$

$${}^t\mathbf{f}_{int} = {}^t\mathbf{B}^T \mathbf{A}_0 {}^t\mathbf{S} \mathbf{L}_0 \quad \text{内力向量}$$

二维两节点杆单元

■ 结构控制方程

$$({}^t\mathbf{K}_M + {}^t\mathbf{K}_G){}^t\Delta\mathbf{u} = {}^{t+\Delta t}\mathbf{f}_{ext} - {}^t\mathbf{f}_{int}$$

其中

$${}^t\mathbf{K}_M = \mathbf{A} \sum_{e=1}^{n_{el}} {}^t\mathbf{K}_M^e, \quad {}^t\mathbf{K}_G = \mathbf{A} \sum_{e=1}^{n_{el}} {}^t\mathbf{K}_G^e$$

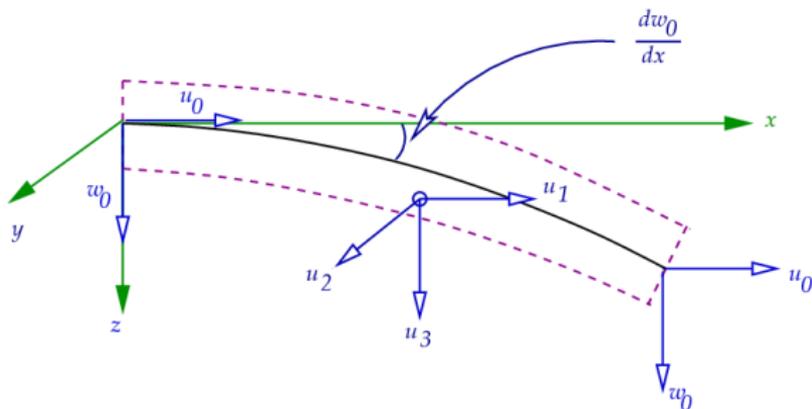
$${}^{t+\Delta t}\mathbf{f}_{ext} = \mathbf{A} \sum_{e=1}^{n_{el}} {}^{t+\Delta t}\mathbf{f}_{ext}^e, \quad {}^t\mathbf{f}_{int} = \mathbf{A} \sum_{e=1}^{n_{el}} {}^t\mathbf{f}_{int}^e$$

结合适当的边界条件

■ 迭代格式

$$\begin{cases} ({}^k\mathbf{K}_M + {}^k\mathbf{K}_G){}^k\Delta\mathbf{u} = {}^{t+\Delta t}\mathbf{f}_{ext} - {}^k\mathbf{f}_{int} \\ {}^{k+1}\mathbf{u} = {}^k\mathbf{u} + {}^k\Delta\mathbf{u} \end{cases}$$

欧拉伯努利梁



$$u_1 = u(X) - z\theta \approx u(X) - z \frac{dw}{dX}, \quad u_2 = 0, \quad u_3 = w(X)$$

虚功原理

■ 虚功原理

$$\int_{\Omega} \boldsymbol{\sigma} \delta \boldsymbol{\epsilon} d\Omega = \int_L f \delta u dx + \int_L m \delta \theta dx + \mathbf{N} \delta \mathbf{u} + \mathbf{Q} \delta \mathbf{w} + \mathbf{M} \delta \boldsymbol{\theta}$$

■ 内力虚功

$$\int_{\Omega} \boldsymbol{\sigma} \delta \boldsymbol{\epsilon} d\Omega_0 = \int_{\Omega_0} S \delta E d\Omega_0 = \int_{L_0} \left(\int_{A_0} S \delta E dA_0 \right) dX$$

应变表达式

- Green 应变

$$E_{IJ} = \frac{1}{2}(u_{I,J} + u_{J,I} + u_{K, I}u_{K, J})$$

- 考虑前述位移表达式，可得 von Karman 应变

$$\begin{aligned} E = E_{11} &= \frac{\partial u}{\partial X} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial X} \right)^2 + \left(\frac{\partial w}{\partial X} \right)^2 \right] \\ &= \frac{du}{dX} - z \frac{d^2 w}{dX^2} + \frac{1}{2} \left[\left(\frac{du}{dX} - z \frac{d^2 w}{dX^2} \right)^2 + \left(\frac{dw}{dX} \right)^2 \right] \\ &\approx \frac{du}{dX} - z \frac{d^2 w}{dX^2} + \frac{1}{2} \left(\frac{dw}{dX} \right)^2 \end{aligned}$$

应变增量

■ 应变增量

$$\Delta E = \underbrace{\frac{d\Delta u}{dX} - z \frac{d^2 \Delta w}{dX^2} + \frac{d\Delta w}{dX} \frac{dw}{dX}}_{\Delta e} + \underbrace{\frac{1}{2} \left(\frac{d\Delta w}{dX} \right)^2}_{\Delta \eta}$$

■ 应变增量变分

$$\delta \Delta E = \underbrace{\frac{d\delta \Delta u}{dX} - z \frac{d^2 \delta \Delta w}{dX^2} + \frac{d\delta \Delta w}{dX} \frac{dw}{dX}}_{\delta \Delta e} + \underbrace{\frac{d\Delta w}{dX} \frac{d\delta \Delta w}{dX}}_{\delta \Delta \eta}$$

- 以 Δu 为自变量： $\Delta e \rightarrow$ 线性， $\Delta \eta \rightarrow$ 二次， $\delta \Delta e \rightarrow$ 常数， $\delta \Delta \eta \rightarrow$ 线性。

内力虚功线性化

- 线弹性应力应变关系

$$\Delta S = C_T \Delta E, \quad S = C_T E$$

- 内力虚功

$$\begin{aligned} \int_{\Omega_0} ({}^{t+\Delta t}S) \delta({}^{t+\Delta t}E) d\Omega_0 &= \int_{\Omega_0} ({}^tS + {}^t\Delta S) \delta({}^tE + {}^t\Delta E) d\Omega_0 \\ &= \int_{\Omega_0} ({}^tS + C_T {}^t\Delta E) \delta({}^t\Delta E) d\Omega_0 \\ &= \underbrace{\int_{\Omega_0} {}^tS \delta^t \Delta e d\Omega_0}_{\text{constant}} + \underbrace{\int_{\Omega_0} {}^tS \delta^t \Delta \eta d\Omega_0 + \int_{\Omega_0} {}^t\Delta e C_T \delta^t \Delta e d\Omega_0}_{\text{linear}} \end{aligned}$$

+ high order terms

内力虚功线性化

■ 第三项

$$\begin{aligned}
 & \int_{\Omega_0} {}^t\Delta e C_T \delta {}^t\Delta e d\Omega_0 \\
 &= \int_{\Omega_0} \left\{ \underbrace{\frac{d^t\Delta u}{dX} + \frac{d^t\Delta w}{dX} \frac{d^t w}{dX}}_{{}^t\Delta e_a} - z \underbrace{\frac{d^2 {}^t\Delta w}{dX^2}}_{{}^t\Delta e_m} \right\} C_T \delta [{}^t\Delta e_a + {}^t\Delta e_m] d\Omega_0 \\
 &= \int_{L_0} {}^t\Delta e_a C_T A_0 \delta {}^t\Delta e_a dL_0 + \int_{L_0} {}^t\Delta e_m C_T I_0 \delta {}^t\Delta e_m dL_0
 \end{aligned}$$

内力虚功线性化

- 第二项

$$\int_{\Omega_0} {}^tS\delta^t\Delta\eta d\Omega_0 = \int_{L_0} \frac{d\Delta w}{dX} \left[\frac{d^t u}{dX} + \frac{1}{2} \left(\frac{d^t w}{dX} \right)^2 \right] C_T A_0 \delta \frac{d\Delta w}{dX} dL_0$$

- 线性化虚功原理化为

$$\begin{aligned} & \int_{L_0} {}^t\Delta e_a {}^tC_T A_0 \delta^t\Delta e_a dL_0 + \int_{L_0} {}^t\Delta e_m {}^tC_T I_0 \delta^t\Delta e_m dL_0 \\ & + \int_{L_0} \frac{d\Delta w}{dX} \left[\frac{d^t u}{dX} + \frac{1}{2} \left(\frac{d^t w}{dX} \right)^2 \right] A_0 \delta \frac{d\Delta w}{dX} dL_0 \\ & = {}^{t+\Delta t}\mathbf{f}_{ext} \delta \mathbf{u} - \int_{\Omega_0} {}^tS\delta^t\Delta e d\Omega_0 \end{aligned}$$

插值函数

- 轴向位移采用线性插值

$$u(X) = \phi_1(X)u_1 + \phi_2(X)u_2$$

- 侧向位移和转角采用厄米插值

$$w(X) = H_1(X)w_1 + H_2(X)\theta_1 + H_3(X)w_2 + H_4(X)\theta_2$$

$$\theta(X) = \frac{dw}{dX}$$

单元方程

- 可得单元矩阵

$$\begin{bmatrix} {}^t\mathbf{K}^{11} & {}^t\mathbf{K}^{12} \\ {}^t\mathbf{K}^{21} & {}^t\mathbf{K}^{22} \end{bmatrix} \begin{bmatrix} {}^t\Delta\mathbf{u} \\ {}^t\Delta\mathbf{d} \end{bmatrix} = \begin{bmatrix} {}^t\Delta\mathbf{F}_u \\ {}^t\Delta\mathbf{F}_d \end{bmatrix}$$

- 其中

$$\mathbf{u} = [u_1, u_2]^T, \quad \mathbf{d} = [w_1, \theta_1, w_2, \theta_2]^T$$

$${}^t\mathbf{K}^{11} \rightarrow 2 \times 2, \quad {}^t\mathbf{K}^{12} \rightarrow 2 \times 4, \quad {}^t\mathbf{K}^{21} \rightarrow 4 \times 2, \quad {}^t\mathbf{K}^{22} \rightarrow 4 \times 4$$

刚度项

$${}^t\mathbf{K}_{ij}^{11} = \int_{L_0} C_T A_0 \frac{d\phi_i}{dX} \frac{d\phi_j}{dX} dL_0$$

$${}^t\mathbf{K}_{ij}^{12} = \int_{L_0} C_T A_0 \frac{d^t w}{dX} \frac{dH_i}{dX} \frac{d\phi_j}{dX} dL_0$$

$${}^t\mathbf{K}_{ij}^{21} = \int_{L_0} C_T A_0 \frac{d^t w}{dX} \frac{d\phi_i}{dX} \frac{dH_j}{dX} dL_0$$

$${}^t\mathbf{K}_{ij}^{22} = \int_{L_0} C_T A_0 \left[\frac{d^t u}{dX} + \frac{1}{2} \left(\frac{d^t w}{dX} \right)^2 \right] \frac{dH_i}{dX} \frac{dH_j}{dX} dL_0$$

$$+ \int_{L_0} C_T I_0 \frac{d^2 H_i}{dX^2} \frac{d^2 H_j}{dX^2} dL_0$$

迭代格式

- 控制方程

$${}^k\mathbb{K} \Delta \mathbf{u} = {}^{t+\Delta t} \mathbf{f}_{ext} - {}^k \mathbf{f}_{int}$$

$${}^k\mathbb{K} = \mathbf{A}_{i=1}^{n_{el}} \begin{bmatrix} {}^k \mathbf{K}^{11} & {}^k \mathbf{K}^{12} \\ {}^k \mathbf{K}^{21} & {}^k \mathbf{K}^{22} \end{bmatrix}_i$$

采用合理的方式引入本质边界条件。

- 迭代

$${}^{k+1} \mathbf{u} = {}^k \mathbf{u} + {}^k \Delta \mathbf{u}$$

轴力（薄膜力）锁死

- 对于轴力接近 0 的情况

$$N = C_T A_0 \left[\frac{du}{dX} + \frac{1}{2} \left(\frac{dw}{dX} \right)^2 \right] \approx 0$$

- 那么轴力增量接近 0

$$\Delta N = C_T A_0 \left[\frac{d\Delta u}{dX} + \frac{dw}{dX} \frac{d\Delta w}{dX} + \frac{1}{2} \left(\frac{d\Delta w}{dX} \right)^2 \right] \approx 0$$

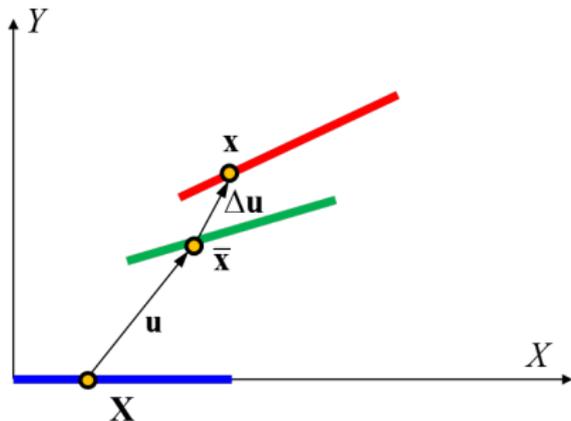
Δu 为线性插值， Δw 厄米（三次）插值。上式中第一项为常数项，第二项为二次项，第三项为四次项，若恒等于 0，则会引起锁闭！

- 解决方法：**选择性减缩积分 (selective reduced integration)**——非线性刚度有关三个积分采用减缩积分。

TL 格式杆单元与梁单元评述

- TL 格式是几何非线性格式中最容易理解和建立控制方程的格式，在实际中也应用最广泛。
- TL 格式适用于一般由大位移和转动引起的几何非线性问题，但是对大变形问题追踪能力一般。
- TL 格式杆单元适用于桁架结构的一般几何非线性问题，且与连续体的几何非线性问题联系紧密，属于同一体系。
- TL 格式梁单元有多种格式，其区别在于对于几何非线性问题的不同简化（或者不做简化，直接建立“几何精确”的格式），这里介绍的单元只适用于**大位移、中等转角、小变形**情况，且还要特别注意其锁闭等问题。梁单元与板壳单元属于同一体系。

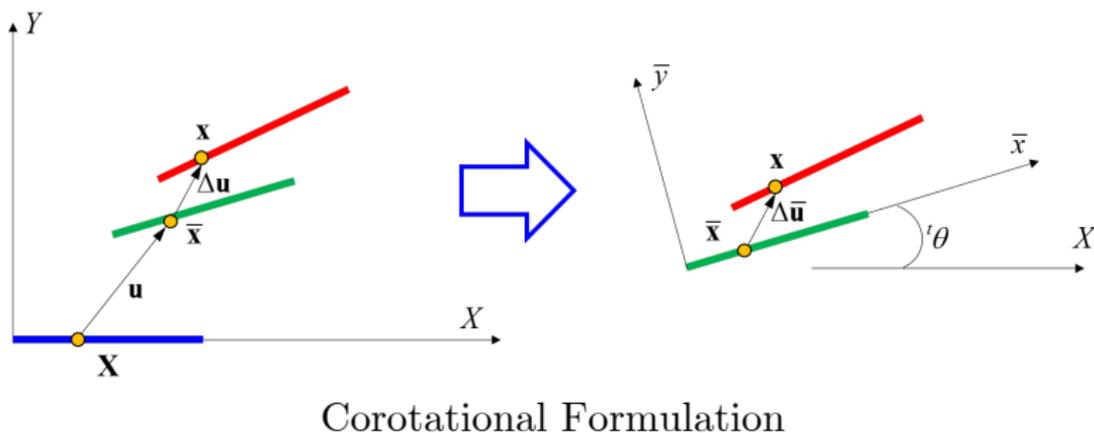
杆系大位移问题



Definition (Update Lagrangian Formulation)

将状态量映射到当前加载步初始点位型，在其上建立控制方程的格式。

随体坐标格式



平衡方程

- 对于变形后状态，杆元满足虚功方程

$$\int_L A \sigma \delta \varepsilon dl = \mathbf{f}_{ext} \delta \mathbf{u}$$

- 映射到 t 时刻的随体坐标系

$$\int_{\bar{L}} \bar{A} \bar{S} \delta \bar{E} d\bar{l} = \bar{\mathbf{f}}_{ext} \delta \bar{\mathbf{u}}$$

- 考虑增量格式

$$\int_{\bar{L}} \bar{A} (\bar{S} + \Delta \bar{S}) \delta \Delta \bar{E} d\bar{l} = \bar{\mathbf{f}}_{ext} \delta \Delta \bar{\mathbf{u}}$$

应变分析

- 格林应变表达式

$$\bar{E} = \bar{E}_{11} = \frac{\partial \bar{u}_1}{\partial \bar{x}_1} + \frac{1}{2} \left[\left(\frac{\partial \bar{u}_1}{\partial \bar{x}_1} \right)^2 + \left(\frac{\partial \bar{u}_2}{\partial \bar{x}_1} \right)^2 \right]$$

- 格林应变表达增量

$$\begin{aligned} \Delta \bar{E} &= \frac{\partial \Delta \bar{u}_1}{\partial \bar{x}_1} + \frac{\partial \bar{u}_1}{\partial \bar{x}_1} \frac{\partial \Delta \bar{u}_1}{\partial \bar{x}_1} + \frac{\partial \bar{u}_2}{\partial \bar{x}_1} \frac{\partial \Delta \bar{u}_2}{\partial \bar{x}_1} \\ &\quad + \frac{1}{2} \left[\left(\frac{\partial \Delta \bar{u}_1}{\partial \bar{x}_1} \right)^2 + \left(\frac{\partial \Delta \bar{u}_2}{\partial \bar{x}_1} \right)^2 \right] \\ &= \underbrace{\frac{\partial \Delta \bar{u}_1}{\partial \bar{x}_1}}_{\Delta \bar{\epsilon}} + \underbrace{\frac{1}{2} \left[\left(\frac{\partial \Delta \bar{u}_1}{\partial \bar{x}_1} \right)^2 + \left(\frac{\partial \Delta \bar{u}_2}{\partial \bar{x}_1} \right)^2 \right]}_{\Delta \bar{\eta}} \end{aligned}$$

应变分析

■ 应变增量变分

$$\delta\Delta\bar{e} = \frac{\partial\Delta\delta\bar{u}_1}{\partial\bar{x}_1}$$

$$\delta\Delta\bar{\eta} = \frac{\partial\Delta\bar{u}_1}{\partial\bar{x}_1} \frac{\partial\Delta\delta\bar{u}_1}{\partial\bar{x}_1} + \frac{\partial\Delta\bar{u}_2}{\partial\bar{x}_1} \frac{\partial\Delta\delta\bar{u}_2}{\partial\bar{x}_1}$$

■ 应力增量

$${}^{t+\Delta t}\bar{S} = {}^t\bar{S} + {}^t\Delta\bar{S} = \bar{\sigma} + C_T\Delta\bar{E}$$

虚功原理

- 将应变、应变表达式代入增量虚功原理可得

$$\int_{\bar{L}} \bar{A} [\bar{\sigma} + C_T (\Delta \bar{e} + \Delta \bar{\eta})] (\delta \Delta \bar{e} + \delta \Delta \bar{\eta}) d\bar{l} = \bar{\mathbf{f}}_{ext} \delta \Delta \bar{\mathbf{u}}$$

- 线性化

$$\begin{aligned} \int_{\bar{L}} \bar{A} \Delta \bar{e} C_T \delta \Delta \bar{e} d\bar{l} + \int_{\bar{L}} \bar{A} \bar{\sigma} \delta \Delta \bar{\eta} d\bar{l} \\ = \bar{\mathbf{f}}_{ext} \delta \Delta \bar{\mathbf{u}} - \int_{\bar{L}} \bar{A} \bar{\sigma} \delta \Delta \bar{e} d\bar{l} \end{aligned}$$

有限元格式

■ 平面两节点杆单元

$$\bar{u}_1 \rightarrow \bar{u}, \bar{u}_2 \rightarrow \bar{v}, \bar{x}_1 \rightarrow \bar{x}, \bar{x}_2 \rightarrow \bar{y}$$

■ 节点位移向量

$$\Delta \bar{\mathbf{u}} = [\Delta \bar{u}_1, \Delta \bar{v}_1, \Delta \bar{u}_2, \Delta \bar{v}_2]^T$$

■ 线性应变

$$\Delta \bar{\epsilon} = \frac{\partial \Delta \bar{u}_1}{\partial \bar{x}_1} = \frac{\Delta \bar{u}_2 - \Delta \bar{u}_1}{\bar{x}_2 - \bar{x}_1} = \frac{\Delta \bar{u}_2 - \Delta \bar{u}_1}{\bar{L}} = \underbrace{\frac{1}{\bar{L}}[-1, 0, 1, 0]}_{\bar{\mathbf{B}}_L} \Delta \bar{\mathbf{u}}$$

$$\delta \Delta \bar{\epsilon} = \bar{\mathbf{B}}_L \Delta \delta \bar{\mathbf{u}}$$

有限元格式

■ 非线性应变

$$\begin{bmatrix} \frac{\partial \Delta \bar{u}_1}{\partial \bar{x}_1} \\ \frac{\partial \Delta \bar{u}_2}{\partial \bar{x}_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial \Delta \bar{u}}{\partial \bar{x}} \\ \frac{\partial \Delta \bar{v}}{\partial \bar{x}} \end{bmatrix} = \begin{bmatrix} \frac{\Delta \bar{u}_2 - \Delta \bar{u}_1}{\bar{x}_2 - \bar{x}_1} \\ \frac{\Delta \bar{v}_2 - \Delta \bar{v}_1}{\bar{x}_2 - \bar{x}_1} \end{bmatrix} = \frac{1}{\bar{L}} \underbrace{\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\mathbf{B}_{NL}} \Delta \bar{\mathbf{u}}$$

$$\delta \Delta \eta = \Delta \bar{\mathbf{u}}^T \mathbf{B}_{NL}^T \mathbf{B}_{NL} \Delta \delta \bar{\mathbf{u}}$$

■ 代入线性化增量虚功方程，单元刚度方程

$$(\bar{\mathbf{K}}_L + \bar{\mathbf{K}}_{NL}) \Delta \bar{\mathbf{u}} = \bar{\mathbf{f}}_{ext} - \bar{\mathbf{f}}_{int}$$

有限元格式

■ 单元刚度方程中各项

$$\bar{\mathbf{K}}_L = \bar{\mathbf{A}}\mathbf{B}_L^T\mathbf{B}_L C_T \bar{\mathbf{L}} = \frac{C_T \bar{\mathbf{A}}}{\bar{L}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\mathbf{K}}_{NL} = \bar{\mathbf{A}}\bar{\sigma}\mathbf{B}_{NL}^T\mathbf{B}_{NL}\bar{\mathbf{L}} = \frac{\bar{\sigma}\bar{\mathbf{A}}}{\bar{L}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\bar{\mathbf{f}}_{int} = \bar{\mathbf{A}}\bar{\sigma}\mathbf{B}_L$$

有限元格式

- 组装结构刚度矩阵之前，须从随体坐标系转换到整体坐标系

$$\begin{bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^t \theta & \sin^t \theta & 0 & 0 \\ -\sin^t \theta & \cos^t \theta & 0 & 0 \\ 0 & 0 & \cos^t \theta & \sin^t \theta \\ 0 & 0 & -\sin^t \theta & \cos^t \theta \end{bmatrix}}_{\bar{T}} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

有限元格式

■ 单元刚度方程各项坐标转换

$$\begin{aligned}
 \mathbf{K}_L &= \bar{\mathbf{T}}^T \bar{\mathbf{K}}_L \bar{\mathbf{T}} \\
 &= \frac{C_T \bar{A}}{\bar{L}} \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta & -\cos^2 \theta & -\cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta & -\cos \theta \sin \theta & -\sin^2 \theta \\ -\cos^2 \theta & -\cos \theta \sin \theta & \cos^2 \theta & \cos \theta \sin \theta \\ -\cos \theta \sin \theta & -\sin^2 \theta & \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}
 \end{aligned}$$

$$\mathbf{K}_{NL} = \bar{\mathbf{T}}^T \bar{\mathbf{K}}_{NL} \bar{\mathbf{T}} = \bar{\mathbf{K}}_{NL}$$

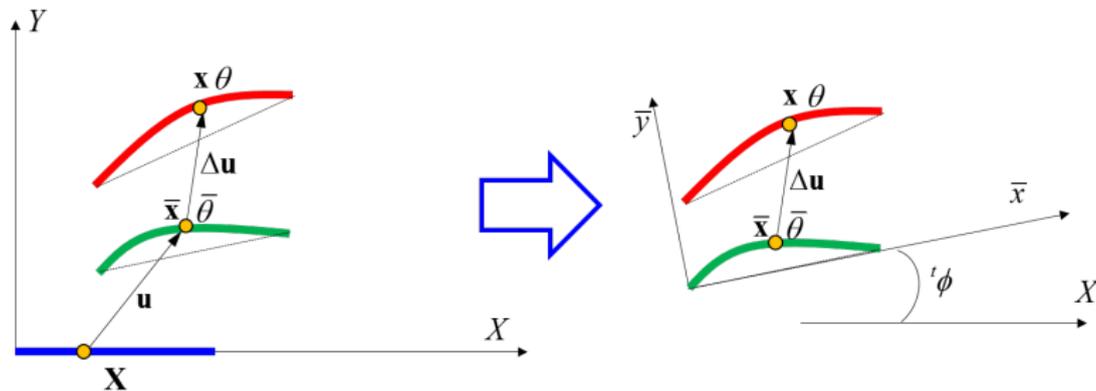
$$\mathbf{f}_{int} = \bar{\mathbf{f}}_{int} \bar{\mathbf{T}} = \bar{A} \bar{\sigma} [-\cos \theta, -\sin \theta, \cos \theta, \sin \theta]^T$$

■ 组装整体刚度矩阵、引入边界条件等.....

UL 格式

- 以当前加载步的初始点为参考状态建立控制方程。
- 整体坐标与随体坐标的转换矩阵与参考状态有关。
- 较之 TL 格式，UL 格式中没有了初始移位项的影响。
- TL 格式与 UL 格式实质上是对同一组控制方程的不同的计算方式。但是，在建立不同格式的过程中为了计算方便，往往会引入一些近似假定，所以实际中二者的计算结果可能会不一样。Bathe 认为，对于路径无关材料，两种格式的误差应该在百分之五以内；对于路径相关材料，二者误差会大一些，此时更倾向于用 UL 格式进行计算。

随体坐标格式



Corotational Formulation

虚功原理

■ 虚功原理

$$\int_{\Omega} \boldsymbol{\sigma} \delta \boldsymbol{\epsilon} d\Omega = \int_L f \delta u dx + \int_L m \delta \theta dx + \mathbf{N} \delta \mathbf{u} + \mathbf{Q} \delta \mathbf{w} + \mathbf{M} \delta \boldsymbol{\theta}$$

■ 内力虚功

$$\int_{\Omega} \boldsymbol{\sigma} \delta \boldsymbol{\epsilon} d\Omega_0 = \int_{\bar{\Omega}} \bar{\boldsymbol{S}} \delta \bar{\mathbf{E}} d\bar{\Omega} \approx \int_{\bar{L}} \left(\int_{\bar{A}} \bar{\boldsymbol{S}} \delta \bar{\mathbf{E}} d\bar{A} \right) d\bar{x}$$

运动方程

- 位移

$$\bar{u}_1 \approx \bar{u}(\bar{x}) - z \frac{d\bar{w}}{d\bar{x}}, \quad \bar{u}_2 = 0, \quad \bar{u}_3 = \bar{w}(\bar{x})$$

- 考虑 Green-von Karman 应变

$$\bar{E} = \bar{E}_{11} \approx \frac{d\bar{u}}{d\bar{x}} - z \frac{d^2\bar{w}}{d\bar{x}^2} + \frac{1}{2} \left(\frac{d\bar{w}}{d\bar{x}} \right)^2$$

- 应变增量

$$\Delta \bar{E} = \underbrace{\frac{d\Delta \bar{u}}{d\bar{x}} - z \frac{d^2\Delta \bar{w}}{d\bar{x}^2} + \frac{d\Delta \bar{w}}{d\bar{x}} \frac{d\bar{w}}{d\bar{x}}}_{\Delta \bar{e}} + \underbrace{\frac{1}{2} \left(\frac{d\Delta \bar{w}}{d\bar{x}} \right)^2}_{\Delta \bar{\eta}}$$

运动方程

■ 应变增量

$$\Delta \bar{E} = \underbrace{\frac{d\Delta \bar{u}}{d\bar{x}} - z \frac{d^2 \Delta \bar{w}}{d\bar{x}^2} + \frac{d\Delta \bar{w}}{d\bar{x}} \frac{d\bar{w}}{d\bar{x}}}_{\Delta \bar{e}} + \underbrace{\frac{1}{2} \left(\frac{d\Delta \bar{w}}{d\bar{x}} \right)^2}_{\Delta \bar{\eta}}$$

■ 应变增量变分

$$\delta \Delta \bar{E} = \underbrace{\frac{d\delta \Delta \bar{u}}{d\bar{x}} - z \frac{d^2 \delta \Delta \bar{w}}{d\bar{x}^2} + \frac{d\delta \Delta \bar{w}}{d\bar{x}} \frac{d\bar{w}}{d\bar{x}}}_{\delta \Delta \bar{e}} + \underbrace{\frac{d\Delta \bar{w}}{d\bar{x}} \frac{d\delta \Delta \bar{w}}{d\bar{x}}}_{\delta \Delta \bar{\eta}}$$

- 对于大位移（转动）、小变形情况，红色部分可以视为 0。

内力虚功线性化

- 线弹性应力应变关系

$$\Delta \bar{S} = C_T \Delta \bar{E}, \quad \bar{S} = C_T \bar{E}$$

- 内力虚功

$$\begin{aligned} \int_{\bar{\Omega}} ({}^{t+\Delta t} \bar{S}) \delta ({}^{t+\Delta t} \bar{E}) d\bar{\Omega} &= \int_{\bar{\Omega}} (\bar{S} + C_T \Delta \bar{E}) \delta (\Delta \bar{E}) d\bar{\Omega} \\ &= \underbrace{\int_{\bar{\Omega}} \bar{S} \delta \Delta \bar{e} d\bar{\Omega}}_{\text{constant}} + \underbrace{\int_{\bar{\Omega}} \bar{S} \delta \Delta \bar{\eta} d\bar{\Omega} + \int_{\bar{\Omega}} \Delta \bar{e} C_T \delta \Delta \bar{e} d\bar{\Omega}}_{\text{linear}} \\ &+ \text{high order terms} \end{aligned}$$

内力虚功线性化

■ 第三项

$$\begin{aligned} \int_{\bar{\Omega}} \Delta \bar{e} C_T \delta \Delta \bar{e} d\bar{\Omega} &= \int_{\bar{\Omega}} \left[\underbrace{\frac{d\Delta \bar{u}}{d\bar{x}}}_{\Delta \bar{e}_a} - z \underbrace{\frac{d^2 \Delta \bar{w}}{d\bar{x}^2}}_{\Delta \bar{e}_m} \right] C_T \delta [\Delta \bar{e}_a + \Delta \bar{e}_m] d\bar{\Omega} \\ &= \int_{\bar{L}} \Delta \bar{e}_a C_T \bar{A} \delta \Delta \bar{e}_a d\bar{L} + \int_{\bar{L}} \Delta \bar{e}_m C_T \bar{I}_0 \delta \Delta \bar{e}_m d\bar{L} \end{aligned}$$

■ 第二项

$$\int_{\bar{\Omega}} \bar{S} \delta \Delta \bar{\eta} d\bar{\Omega} = \int_{\bar{L}} \frac{d\Delta \bar{w}}{d\bar{x}} \left[\frac{d\bar{u}}{d\bar{x}} + \frac{1}{2} \left(\frac{d\bar{w}}{d\bar{x}} \right)^2 \right] C_T \bar{A} \delta \frac{d\Delta \bar{w}}{d\bar{x}} d\bar{L}$$

内力虚功线性化

- 线性化虚功原理化为

$$\begin{aligned}
 & \int_{\bar{L}} \Delta \bar{e}_a C_T \bar{A} \delta \Delta \bar{e}_a d\bar{L} + \int_{\bar{L}} \Delta \bar{e}_m C_T \bar{I}_0 \delta \Delta \bar{e}_m d\bar{L} \\
 & + \int_{\bar{L}} \frac{d\Delta \bar{w}}{d\bar{x}} \left[\frac{d\bar{u}}{d\bar{x}} + \frac{1}{2} \left(\frac{d\bar{w}}{d\bar{x}} \right)^2 \right] C_T \bar{A} \delta \frac{d\Delta \bar{w}}{d\bar{x}} d\bar{L} \\
 & = {}^{t+\Delta t} \mathbf{f}_{ext} \delta \mathbf{u} - \int_{\bar{\Omega}} \bar{S} \delta \Delta \bar{e} d\Omega_0
 \end{aligned}$$

插值函数

- 轴向位移采用线性插值

$$\bar{u}(\bar{x}) = \phi_1(\bar{x})\bar{u}_1 + \phi_2(\bar{x})\bar{u}_2$$

- 侧向位移和转角采用厄米插值

$$\bar{w}(\bar{x}) = H_1(\bar{x})\bar{w}_1 + H_2(\bar{x})\bar{\theta}_1 + H_3(\bar{x})\bar{w}_2 + H_4(\bar{x})\bar{\theta}_2, \quad \bar{\theta}(\bar{x}) = \frac{d\bar{w}}{d\bar{x}}$$

- 可得随体坐标系下单元矩阵

$$\begin{bmatrix} \bar{\mathbf{K}}^{11} & \\ & \bar{\mathbf{K}}^{22} \end{bmatrix} \begin{bmatrix} \Delta \bar{\mathbf{u}} \\ \Delta \bar{\mathbf{d}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{F}}_u \\ \bar{\mathbf{F}}_d \end{bmatrix}$$

$$\bar{\mathbf{u}} = [\bar{u}_1, \bar{u}_2]^T, \quad \bar{\mathbf{d}} = [\bar{w}_1, \bar{\theta}_1, \bar{w}_2, \bar{\theta}_2]^T, \quad \bar{\mathbf{K}}^{11} \rightarrow 2 \times 2, \quad \bar{\mathbf{K}}^{22} \rightarrow 4 \times 4$$

刚度项

■ 单元刚度矩阵对应项

$$\bar{\mathbf{K}}_{ij}^{11} = \int_{\bar{L}} C_T \bar{A} \frac{d\phi_i}{d\bar{x}} \frac{d\phi_j}{d\bar{x}} d\bar{L}$$

$$\begin{aligned} \bar{\mathbf{K}}_{ij}^{22} = & \int_{\bar{L}} C_T \bar{A} \left[\frac{d\bar{u}}{d\bar{x}} + \frac{1}{2} \left(\frac{d\bar{w}}{d\bar{x}} \right)^2 \right] \frac{dH_i}{d\bar{x}} \frac{dH_j}{d\bar{x}} d\bar{L} \\ & + \int_{\bar{L}} C_T \bar{I}_0 \frac{d^2 H_i}{d\bar{x}^2} \frac{d^2 H_j}{d\bar{x}^2} d\bar{L} \end{aligned}$$

■ 整体坐标下单元刚度矩阵换算

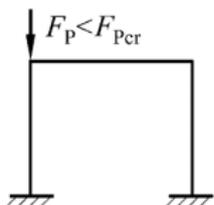
$$\mathbf{K} = \bar{\mathbf{T}}^T \bar{\mathbf{K}} \bar{\mathbf{T}}$$

坐标转换矩阵

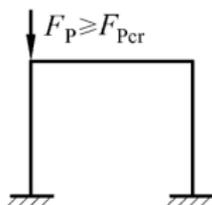
- 随体坐标系到整体坐标系坐标转换矩阵

$$\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^t \phi & 0 & \sin^t \phi & 0 & 0 & 0 \\ 0 & \cos^t \phi & 0 & 0 & \sin^t \phi & 0 \\ -\sin^t \phi & 0 & \cos^t \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sin^t \phi & 0 & 0 & \cos^t \phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\bar{T}} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

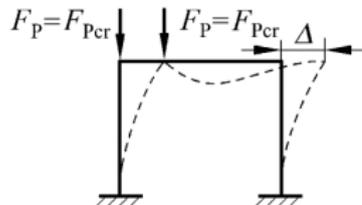
结构稳定问题



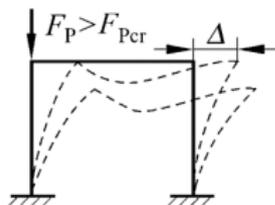
(a) 无弯曲变形



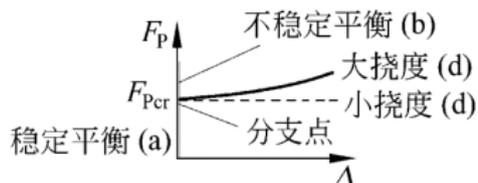
(b) 无任何干扰



(c) 受微小干扰



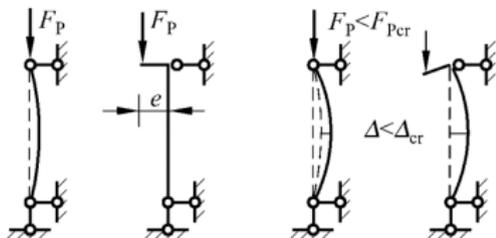
(d) 受微小干扰结构可能破坏



(e) 分支点失稳力和位移的关系

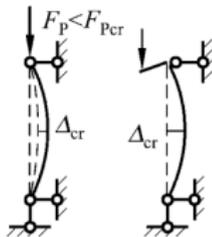
第一类失稳问题 (分支点失稳)

结构稳定问题

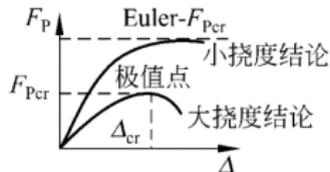


(a) 非完善体系

(b) 稳定平衡状态

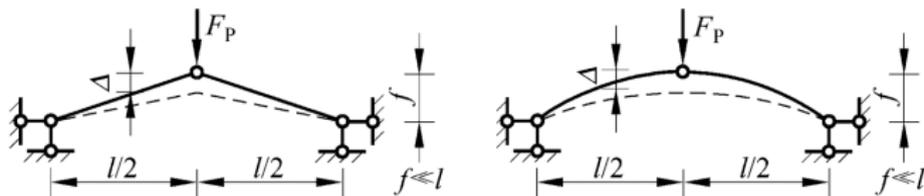


(c) 极限状态, 受干扰不稳定

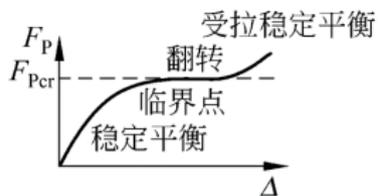
(d) 极值点失稳
力和位移的关系

第二类失稳问题 (极值点失稳)

结构稳定问题



(a) 扁平桁架和扁平拱

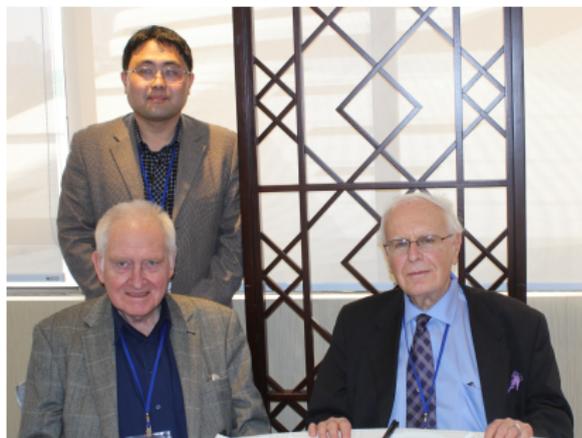


(b) 急跳现象力和位移关系

第三失稳问题（其它失稳问题，如突跳、急回等）

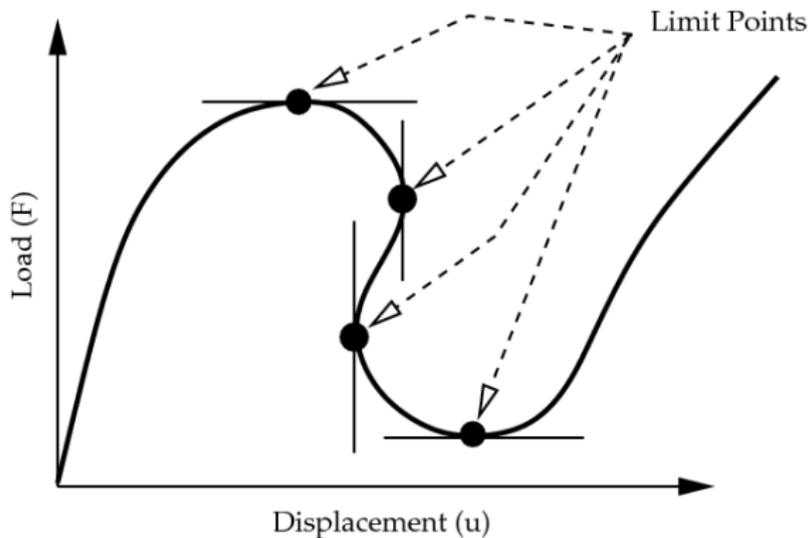
结构稳定问题

- The definition of stability is unstable! (Z.P. Bazant)



- 第二、三类失稳问题一般采用连续等追踪方法分析。
- 第一类失稳问题一般采用线性摄动 (linear perturbation) 方法分析。

连续追踪方法



连续追踪失稳点（基于切线刚度矩阵）

线性摄动方法

- 分支点之前，结构处于线弹性小变形状态。
- 分支点之后，结构进入几何非线性状态，并发生失稳。
- 将分支点作为 t 状态，忽略初始弯曲、偏心等因素，此时对 **应挠度 w 及其对坐标的导数为 0**，在此条件下可大大简化相关表达式和分析过程。
- 分析较为简单，很多情况可以得到解析结果，但是高估了一般构件、结构的失稳荷载。
- 在上述条件下，针对（包含了轴力效应的）刚度矩阵进行分析，得到失稳荷载。
- 工程设计公式往往是线性摄动计算结果乘以折减系数得到。

刚度项 (TL 格式)

$${}^t\mathbf{K}_{ij}^{11} = \int_{L_0} C_T A_0 \frac{d\phi_i}{dX} \frac{d\phi_j}{dX} dL_0$$

$${}^t\mathbf{K}_{ij}^{12} = \int_{L_0} C_T A_0 \frac{d^t w}{dX} \frac{dH_i}{dX} \frac{d\phi_j}{dX} dL_0 = 0, \quad {}^t\mathbf{K}_{ij}^{21} = \int_{L_0} C_T A_0 \frac{d^t w}{dX} \frac{d\phi_i}{dX} \frac{dH_j}{dX} dL_0 = 0$$

$$\begin{aligned} {}^t\mathbf{K}_{ij}^{22} &= \int_{L_0} C_T A_0 \left[\frac{d^t u}{dX} + \frac{1}{2} \left(\frac{d^t w}{dX} \right)^2 \right] \frac{dH_i}{dX} \frac{dH_j}{dX} dL_0 + \int_{L_0} C_T I_0 \frac{d^2 H_i}{dX^2} \frac{d^2 H_j}{dX^2} dL_0 \\ &= \underbrace{\int_{L_0} {}^t N \frac{dH_i}{dX} \frac{dH_j}{dX} dL_0}_{\text{stiffness of axial force}} + \int_{L_0} C_T I_0 \frac{d^2 H_i}{dX^2} \frac{d^2 H_j}{dX^2} dL_0 \end{aligned}$$

$${}^t\mathbb{K} = \mathbf{A}_{i=1}^{n_{el}} \begin{bmatrix} {}^t\mathbf{K}^{11} & \\ & {}^t\mathbf{K}^{22} \end{bmatrix}_i, \quad |{}^t\mathbb{K}| = 0 \Rightarrow P_{cr}$$

结束

