有限单元法研究生核心课程

第九讲 梁单元

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期末大作业

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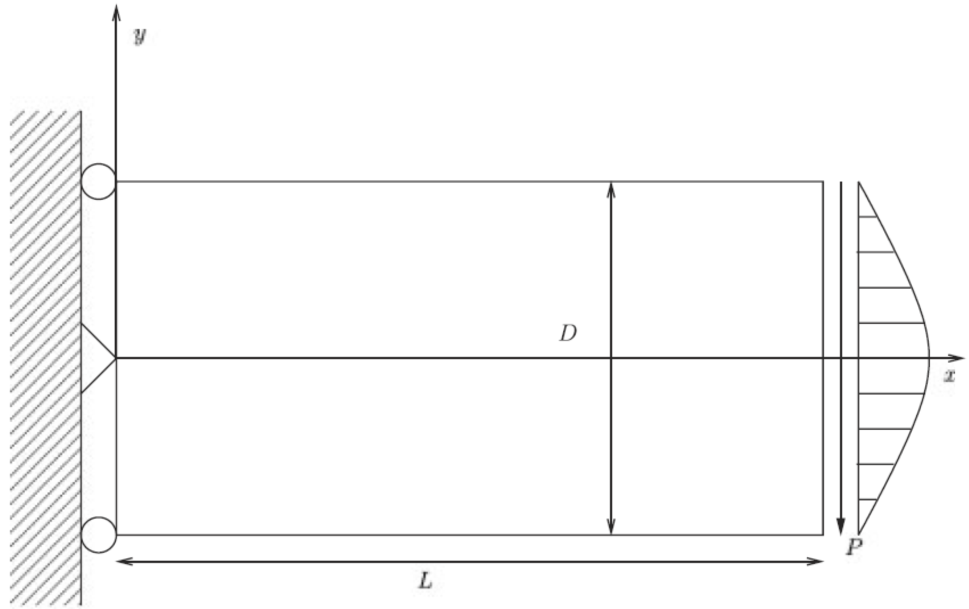

二维梁的信息如下：L=8，D=4，厚度t=1，弹性模量E=3.e7，泊松比v=0.2，荷载P=200。试完成以下计算：

1. **自行编制二维有限元程序，采用平面四节点单元，同时考虑全积分和减缩积分，模拟梁受力之后的位移和应力。**
2. **针对不同数值积分方式所得计算结果，变化合适的高宽比（1:2~1:10），讨论不同数值积分方式可能引起的锁闭、沙漏等问题。**
3. **在双对数坐标中画出采用减缩积分单元所得计算所得位移、应力结果的误差范数曲线。**

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提示：

1. **梁的位移和内力的解析解如下：**

$$u_x = \frac{Py}{6EI}\left[(6L-3x)x+(2+\nu)\left(y^2-\frac{D^2}{4}\right)\right]$$

$$u_y = -\frac{P}{6EI}\left[3\nu y^2(L-x)+(4+5\nu)\frac{D^2}{4}x+(3L-x)x^2\right]$$

$$\sigma_{xx} = \frac{P(L-x)y}{I}$$

$$\sigma_{yy} = 0$$

$$\tau_{xy} = \frac{P}{2I}\left(y^2-\frac{D^2}{4}\right)$$

$$I = \frac{tD^3}{12}$$

2. **误差范数的计算公式如下，计算中须至少变化5个单元长度，最大单元长度与最小单元长度之间跨过3个长度量级，采用比有限元计算过程中更加精细的数值积分，才可能得到稳定且成一条直线的结果。**

$$\left\| f-\hat{f} \right\|_2 = \left[\int_\Omega \left(f-\hat{f}\right)^2 \mathrm{d}\Omega\right]^{\frac{1}{2}}$$

3. **须根据解析解精确施加边界条件，不能随意采用圣维南原理，否则误差范数计算将会出现问题。**

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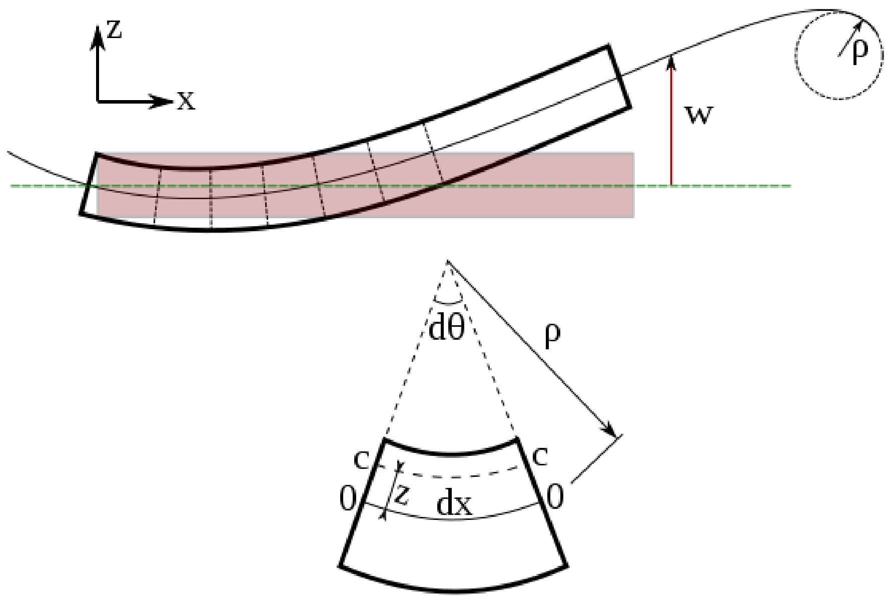
工程梁单元



$$\varepsilon_y = 0, \varepsilon_{xy} = 0$$

$$\tilde{u} = u - z\theta \qquad \theta = \frac{\partial w}{\partial x}$$

$$\varepsilon_x = \frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$$

$$\sigma_x = E\varepsilon_x = E\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)$$

Euler-Bernoulli梁理论

工程梁单元

$$\sigma_x = E\varepsilon_x = E\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)$$

$$N = \int_A \sigma_x dA = EA\frac{\partial u}{\partial x} \xRightarrow{\frac{\partial N}{\partial x} = f(x)} EA\frac{\partial^2 u}{\partial x^2} = f(x)$$

$$M = \int_A \sigma_x z dA = -EI\frac{\partial^2 w}{\partial x^2} \xRightarrow[\frac{\partial Q}{\partial x} = -q(x)]{\frac{\partial M}{\partial x} = Q} \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = q(x)$$

轴向变形与横向变形互不影响，可分别构造求解方法。

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轴向变形控制方程/泛函：

$$EA\frac{\partial^2 u}{\partial x^2}=f(x) \qquad \Pi=\int_L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx-\int_L f(x)u\,dx-\sum_j P_j u_j$$

弱形式控制方程：$\int_L \frac{d\delta u}{dx} EA\frac{du}{dx}dx-\int_L f(x)\delta u\,dx-\sum_j P_j\delta u_j=0$

按照单元积分：

$$\mathop{\mathbf{A}}_{e=1}^{n_{el}}\int_{L^e} \frac{d\delta u}{dx} EA\frac{du}{dx}dx-\mathop{\mathbf{A}}_{e=1}^{n_{el}}\int_{L^e} f(x)\delta u\,dx-\sum_j F_j\delta u_j=0$$

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插值： $$u = \sum_{i=1}^{n} N_i(\xi) u_i = \mathbf{N}\mathbf{u}^e$$

两节点单元： $$N_1(\xi) = 1 - \xi \qquad N_2(\xi) = \xi \qquad 0 \le \xi \le 1$$

$$\mathbf{Kd} = \mathbf{P}$$

$$\mathbf{K} = \mathop{\mathsf{A}}_{e=1}^{n_{el}} \mathbf{K}^e, \mathbf{K}^e = \int_0^1 \frac{EA}{l} \left(\frac{d\mathbf{N}}{d\xi} \right)^{\mathrm{T}} \frac{d\mathbf{N}}{d\xi} d\xi$$

$$\mathbf{K}^e = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{P} = \mathop{\mathsf{A}}_{e=1}^{n_{el}} \mathbf{P}^e, \mathbf{P}^e = \int_0^1 (\mathbf{N})^{\mathrm{T}} f(\xi) l d\xi + \sum_j \mathbf{N}^{\mathrm{T}}(\xi_j) P_j$$

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侧向变形控制方程/泛函：

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)=q(x)$$

$$\Pi=\int_L \frac{EI}{2}\left(\frac{d^2 w}{dx^2}\right)^2 dx-\int_L q(x)w\,dx-\sum_j P_j w_j+\sum_k M_k\theta_k$$

弱形式控制方程：

$$\int_L \frac{d^2\delta w}{dx^2}EI\frac{d^2 w}{dx^2}dx-\int_L q(x)\delta w\,dx-\sum_j P_j\delta w_j+\sum_k M_k\delta\theta_k=0$$

按照单元积分：

$$\mathop{\mathbf{A}}_{e=1}^{n_{el}}\int_{L^e}\frac{d^2\delta w}{dx^2}EI\frac{d^2 w}{dx^2}dx-\mathop{\mathbf{A}}_{e=1}^{n_{el}}\int_{L^e}q(x)\delta w\,dx-\sum_j P_j\delta w_j+\sum_k M_k\delta\theta_k=0$$

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$$w(\xi) = \sum_{i=1}^{2} H_i^{(0)}(\xi) w_i + \sum_{i=1}^{2} H_i^{(1)}(\xi)\theta_i = \mathbf{N}\mathbf{a}^e \qquad \mathbf{a}^e = \left[w_1, \theta_1, w_2, \theta_2\right]$$

$$N_3 = 1 - 3\xi^2 + 2\xi^3, N_4 = (\xi - 2\xi^2 + \xi^3)l, N_5 = 3\xi^2 - 2\xi^3, N_6 = (-\xi^2 + \xi^3)l$$

$$0 \le \xi \le 1$$

$$\mathbf{Ka} = \mathbf{P}$$

$$\mathbf{K} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \mathbf{K}^e, \mathbf{P} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \mathbf{P}^e$$

$$\mathbf{K}^e = \int_0^1 \frac{EI}{l^3} \left(\frac{d^2\mathbf{N}}{d\xi^2} \right)^{\mathrm{T}} \frac{d^2\mathbf{N}}{d\xi^2} d\xi$$

$$\mathbf{K}^e = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\mathbf{P}^e = \int_0^1 (\mathbf{N})^{\mathrm{T}} q(\xi) l d\xi + \sum_j \mathbf{N}^{\mathrm{T}}(\xi_j) P_j - \sum_k M_k \frac{d\mathbf{N}^{\mathrm{T}}(\xi_k)}{l d\xi}$$

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组合杆单元与梁单元

$$\mathbf{a}_i=[u_i,w_i,\theta_i]^T \qquad \mathbf{F}_i=[N_i,P_i,M_i]^T$$

插值矩阵

$$\begin{Bmatrix} u \\ v \\ \theta \end{Bmatrix}=\mathbf{Na}^e=\begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_3 & N_4 & 0 & N_5 & N_6 \\ 0 & 0 & N_1 & 0 & 0 & N_2 \end{bmatrix}\begin{bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{bmatrix}$$

单元刚度矩阵

（两端刚接）

$$\boldsymbol{K}^e=\begin{bmatrix} EA/l & 0 & 0 & -EA/l & 0 & 0 \\ & 12EI/l^3 & 6EI/l^2 & 0 & -12EI/l^3 & 6EI/l^2 \\ & & 4EI/l & 0 & -6EI/l^2 & 2EI/l \\ & \text{对} & & EA/l & 0 & 0 \\ & & \text{称} & & 12EI/l^3 & -6EI/l^2 \\ & & & & & 4EI/l \end{bmatrix}$$

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单元刚度矩阵

（一端刚接一端铰接）

$$
\boldsymbol{K}^e = \begin{bmatrix}
EA/l & 0 & 0 & -EA/l & 0 & 0 \\
 & 3EI/l^3 & 3EI/l^2 & 0 & -3EI/l^3 & 0 \\
 & & 3EI/l & 0 & -3EI/l^2 & 0 \\
 & \text{对} & & EA/l & 0 & 0 \\
 & & \text{称} & & 3EI/l^3 & 0 \\
 & & & & & 0
\end{bmatrix}
$$

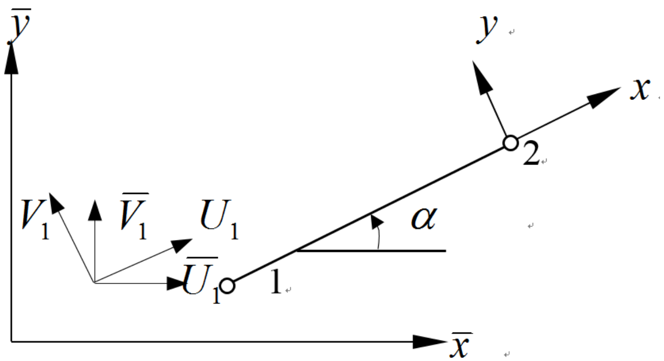
单元刚度矩阵

（两端铰接）

$$
\boldsymbol{K}^e = \begin{bmatrix}
EA/l & 0 & 0 & -EA/l & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 \\
 & \text{对} & & EA/l & 0 & 0 \\
 & & \text{称} & & 0 & 0 \\
 & & & & & 0
\end{bmatrix}
$$

工程梁单元

坐标转换



$$\mathbf{F}^e = \mathbf{T}\bar{\mathbf{F}}^e$$

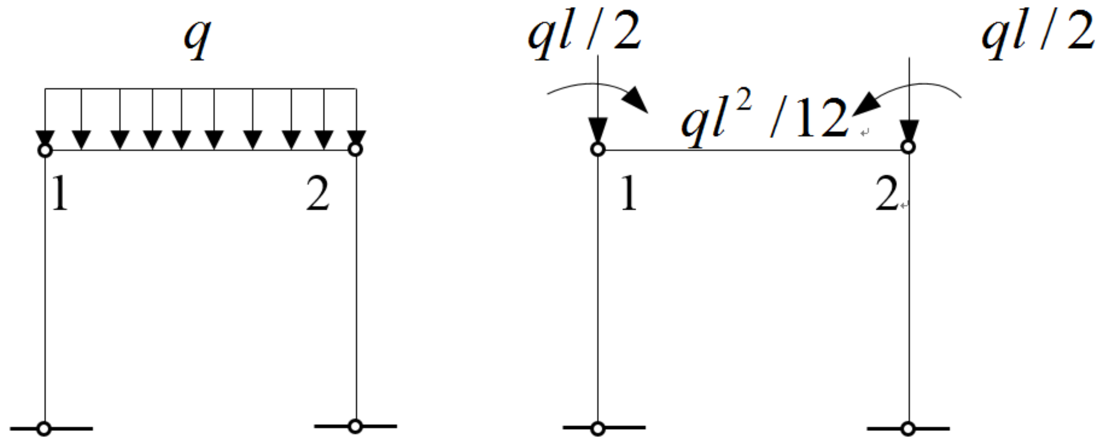
$$\mathbf{a}^e = \mathbf{T}\bar{\mathbf{a}}^e$$

$$\bar{\mathbf{K}}^e = \mathbf{T}^{\mathrm{T}}\mathbf{K}^e\mathbf{T}$$

$$T = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

工程梁单元

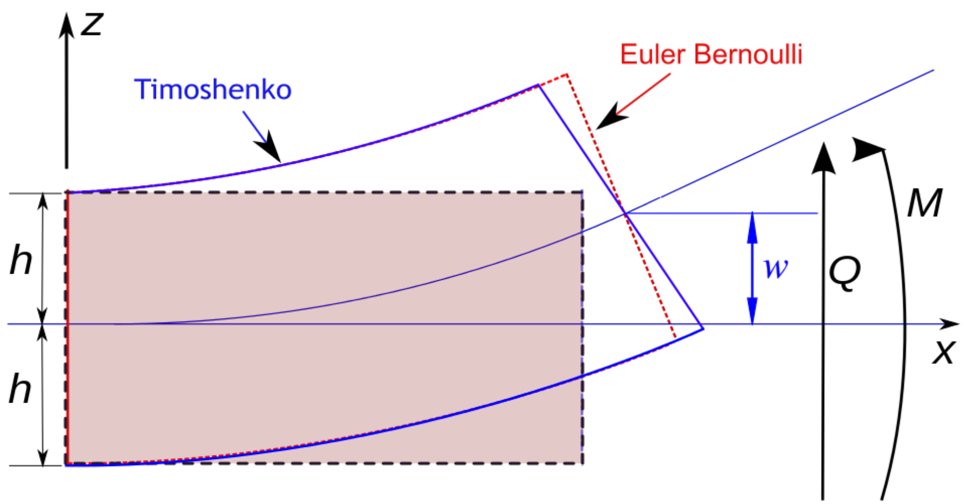
等效节点荷载



$$\mathbf{P}^e = \int_0^1 (\mathbf{N})^{\mathrm{T}} q(\xi) l d\xi$$

支座约束与支座位移：罚函数法！

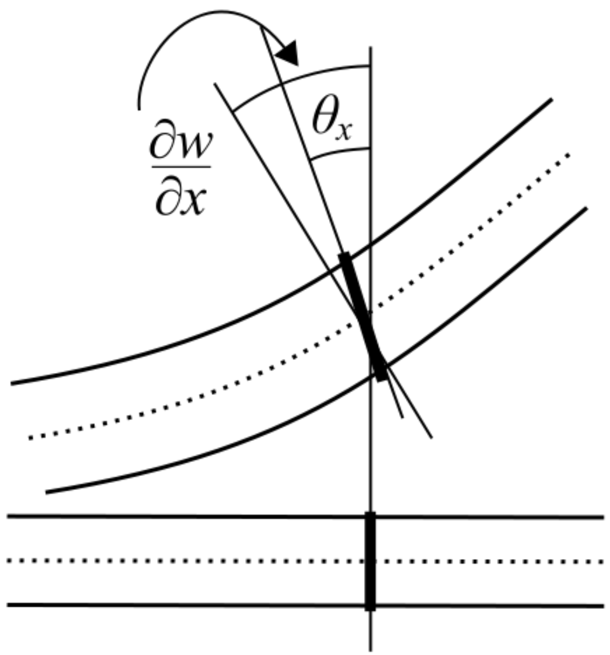
剪切梁单元



Timoshenko梁理论

Timoshenko medal

剪切梁单元



$$\gamma = \frac{dw}{dx} - \theta$$

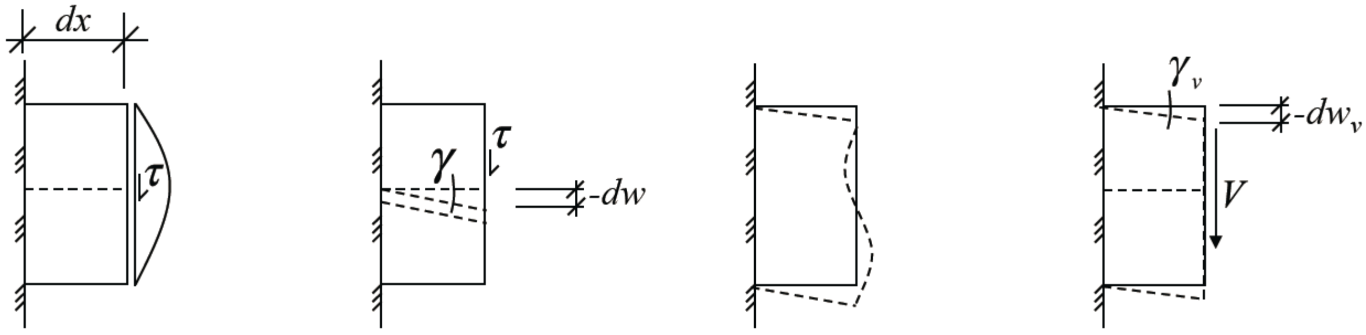
先不考虑轴向变形：

$$u = -z\theta(x), v = 0, w = w(x)$$

$$M = -EI\frac{\mathrm{d}\theta}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = q(x)$$

剪切梁单元



(a) Shear stress　(b) Fibre deformation　(c) Actual shear deformation　(d) Average shear deformation

$$\gamma V=\int_A \tilde{\gamma}\tau dA=\int_A \frac{\tau^2}{G}dA=\int_A \frac{1}{G}\left(\frac{VS^*}{It}\right)^2 dA$$

定义：$\gamma=\dfrac{V}{k_s AG}$　　$k_s=\dfrac{I^2}{A\int_A \left(S^*/t\right)^2 dA}$

剪切梁单元

$$\frac{dw}{dx}-\theta=\frac{1}{k_s AG}\frac{\mathrm{d}M}{\mathrm{d}x}=\frac{1}{k_s AG}\frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)=q(x)$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}-\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{1}{k_s AG}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)=\frac{q(x)}{k_s AG}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}+\frac{q(x)}{k_s AG}\right)\right]=q(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)=q(x)-\frac{EI}{k_s AG}\frac{\mathrm{d}^2 q(x)}{\mathrm{d}x^2}$$

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能量泛函：

$$\Pi = \int_L \frac{EI}{2}\left(\frac{d\theta}{dx}\right)^2 dx + \int_L \frac{Gk_s A}{2}\gamma^2 dx - \int_L q(x) w dx - \sum_j P_j w_j + \sum_k M_k \theta_k$$

弱形式：

$$\int_L \frac{d\delta\theta}{dx} EI \frac{d\theta}{dx} dx + \int_L \left(\frac{d\delta w}{dx} - \delta\theta\right) Gk_s A \left(\frac{dw}{dx} - \theta\right) dx - \int_L q(x)\delta w dx - \sum_j P_j \delta w_j + \sum_k M_k \delta\theta_k = 0$$

单元积分：

$$\mathop{\mathbf{A}}_{e=1}^{n_{el}} \int_{L^e} \frac{d\delta\theta}{dx} EI \frac{d\theta}{dx} dx + \mathop{\mathbf{A}}_{e=1}^{n_{el}} \int_{L^e} \left(\frac{d\delta w}{dx} - \delta\theta\right) Gk_s A \left(\frac{dw}{dx} - \theta\right) dx - \mathop{\mathbf{A}}_{e=1}^{n_{el}} \int_{L^e} q(x)\delta w dx - \sum_j P_j \delta w_j + \sum_k M_k \delta\theta_k = 0$$

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剪切梁单元

位移、转角分别插值（C0连续性，Timoshenko梁单元）：

$$w=\sum_{i=1}^{n}N_i w_i \qquad \theta=\sum_{j=1}^{n}N_j\theta_j$$

$$\mathbf{Ka}=\mathbf{P} \qquad \mathbf{K}=\mathop{\mathsf{A}}_{e=1}^{n_{el}}\mathbf{K}^e,\ \mathbf{P}=\mathop{\mathsf{A}}_{e=1}^{n_{el}}\mathbf{P}^e$$

$$\mathbf{K}^e=\int_0^1\mathbf{B}_b^T EI\mathbf{B}_b l\,d\xi+\int_0^1\mathbf{B}_s^T Gk_s A\mathbf{B}_s l\,d\xi$$

$$\mathbf{B}_b^T=\begin{bmatrix}0 & 0 & \cdots & 0\\ \dfrac{dN_1}{dx} & \dfrac{dN_2}{dx} & \cdots & \dfrac{dN_n}{dx}\end{bmatrix}^T \qquad \mathbf{B}_b^T=\begin{bmatrix}\dfrac{dN_1}{dx} & \dfrac{dN_2}{dx} & \cdots & \dfrac{dN_n}{dx}\\ -N_1 & -N_2 & \cdots & -N_n\end{bmatrix}^T$$

$$\mathbf{P}^e=\int_0^1(\mathbf{N})^{\mathrm{T}}\begin{bmatrix}q(\xi)\\0\end{bmatrix}l\,d\xi+\sum_j\mathbf{N}^{\mathrm{T}}(\xi_j)\begin{bmatrix}P_j\\0\end{bmatrix}-\sum_k\mathbf{N}^{\mathrm{T}}(\xi_k)\begin{bmatrix}0\\M_k\end{bmatrix}$$

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剪切梁单元

能量泛函： $\Pi = \int_L \frac{EI}{2}\left(\frac{d\theta}{dx}\right)^2 dx + \int_L \frac{Gk_s A}{2}\left(\frac{dw}{dx}-\theta\right)^2 dx$

两节点一阶线性插值： $w = \sum_{i=1}^{2} N_i w_i \quad \theta = \sum_{i=1}^{2} N_i \theta_i$

$$\Rightarrow \quad \gamma = \frac{dw}{dx} - \theta = \frac{1}{l}(w_2 - w_1) - \frac{1}{2}(\theta_1 + \theta_2) - \frac{1}{2}(\theta_1 - \theta_2)\xi$$

对于浅梁，系数成为
被动罚数！

$$\xrightarrow{\gamma \to 0} \quad \frac{1}{l}(w_2 - w_1) = \frac{1}{2}(\theta_1 + \theta_2) \qquad \theta_1 - \theta_2 = 0$$

锁死！

减缩积分：1点高斯积分，只在ξ=0处取值

剪切梁单元

一般插值：

$$w=\sum_{i=1}^{n_w} N_i w_i \qquad \theta=\sum_{i=1}^{n_\theta} N_i \theta_i$$

剪切应变：

$$\gamma=\frac{dw}{dx}-\theta=\sum_{i=1}^{n_w}\frac{\mathrm{d}N_i}{\mathrm{d}x}w_i-\sum_{i=1}^{n_\theta}N_i\theta_i$$

抛弃 $\gamma\rightarrow 0$ 可能引起锁死的项

$$\tilde{\gamma}=\sum_{i=1}^{\bar{n}_w}\tilde{B}_i w_i-\sum_{i=1}^{\bar{n}_\theta}\tilde{N}_i\theta_i\approx\gamma$$

直接代入弱形式计算：

$$\mathop{\mathbf{A}}_{e=1}^{n_{el}}\int_{L^e}\frac{d\delta\theta}{dx}EI\frac{d\theta}{dx}dx+\mathop{\mathbf{A}}_{e=1}^{n_{el}}\int_{L^e}\delta\tilde{\gamma}Gk_sA\tilde{\gamma}dx-\mathop{\mathbf{A}}_{e=1}^{n_{el}}\int_{L^e}q(x)\delta w dx-\sum_j P_j\delta w_j+\sum_k M_k\delta\theta_k=0$$

假设剪切应变方法！

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今天就到这里，

明天的事儿明天再说！

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