有限单元法研究生核心课程

第四讲 单元与插值构造

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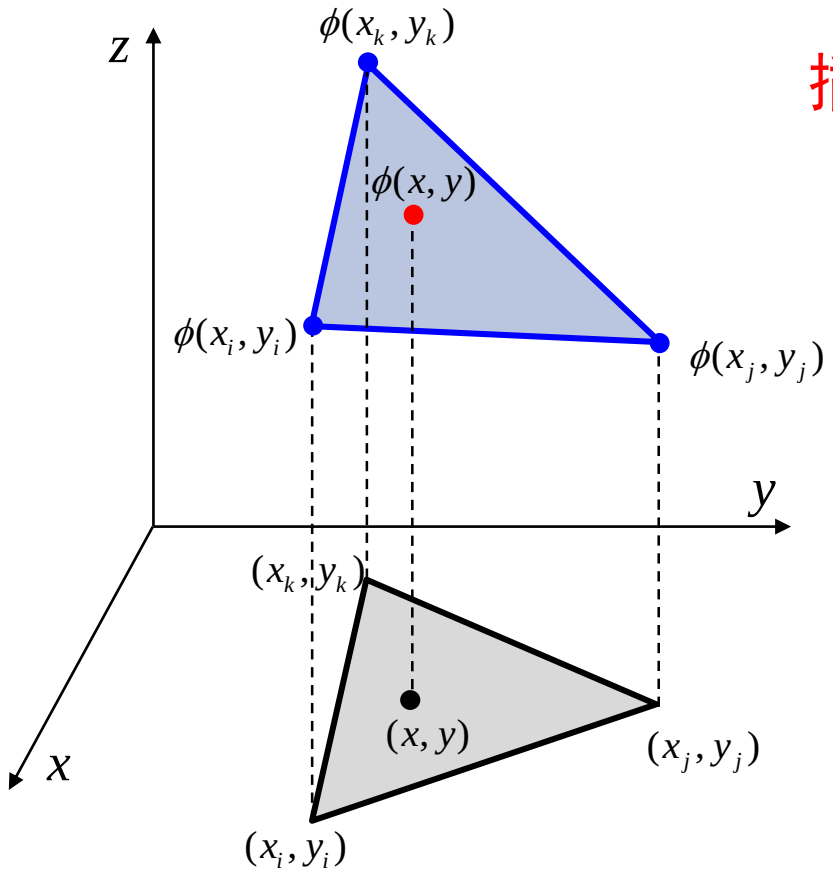
平面三角形单元

$$\phi(x,y)=N_i(x,y)\Phi_i+N_j(x,y)\Phi_j+N_k(x,y)\Phi_k$$



插值函数：

$$\begin{cases} N_i(x,y)=\dfrac{1}{2A}\left(a_i+b_ix+c_iy\right) \\ N_j(x,y)=\dfrac{1}{2A}\left(a_j+b_jx+c_jy\right) \\ N_k(x,y)=\dfrac{1}{2A}\left(a_k+b_kx+c_ky\right) \end{cases}$$

Kronecker delta $N_i(x_j,y_j)=\delta_{ij}$

Partition of unity $N_i(x,y)+N_j(x,y)+N_k(x,y)=1$

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一维单元

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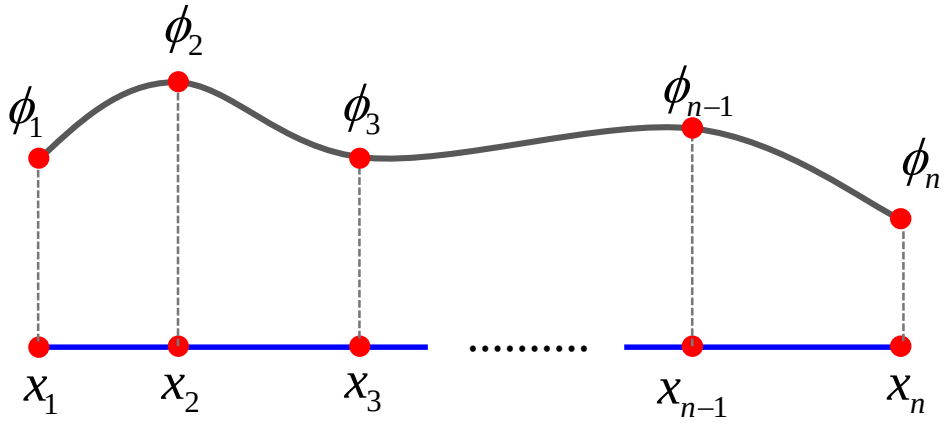
拉格朗日单元



$$\phi(x)=\sum_{i=1}^{n}N_i(x)\phi_i$$

Kronecker delta $N_i(x_j)=\delta_{ij}$

$$N_i(x)=l_i^{(n-1)}(x)=\prod_{j=1,j\neq i}^{n}\frac{x-x_j}{x_i-x_j}$$

Partition of unity $\sum_{i=1}^{n}N_i(x)=1$

拉格朗日单元内部可构造高阶连续的插值函数，
但是单元间始终为C0连续性！

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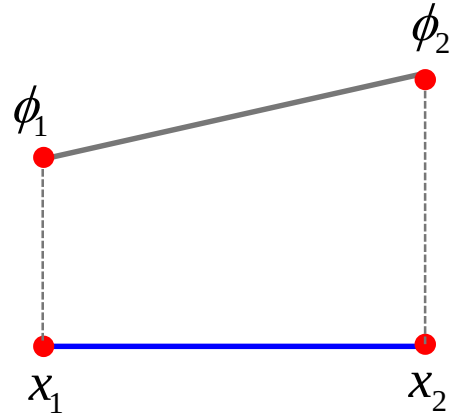
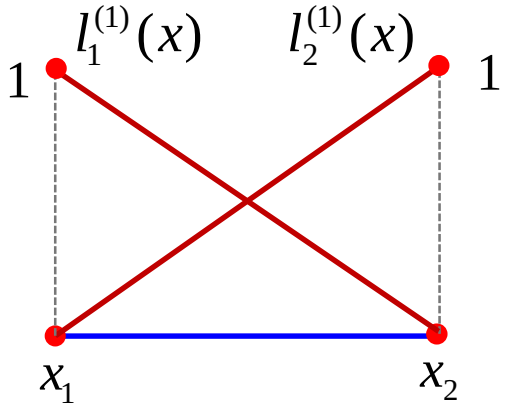
拉格朗日单元



$$\phi(x)=\sum_{i=1}^{2} l_i^{(1)}(x)\phi_i$$

$$l_1^{(1)}(x)=\frac{x-x_2}{x_1-x_2}$$

$$l_2^{(1)}(x)=\frac{x-x_1}{x_2-x_1}$$



一般桁架单元！

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拉格朗日单元

引入无量纲单元坐标（自然坐标）

$$\xi = \frac{x - x_1}{x_n - x_1}$$

$$l_i^{(n-1)}(\xi) = \prod_{j=1, j\neq i}^{n} \frac{\xi - \xi_j}{\xi_i - \xi_j} = \prod_{j=1, j\neq i}^{n} \frac{f_j(\xi)}{f_j(\xi_i)}$$

Kronecker delta $N_i(\xi_j) = \delta_{ij}$

Partition of unity $\sum_{i=1}^{n} N_i(\xi) = 1$

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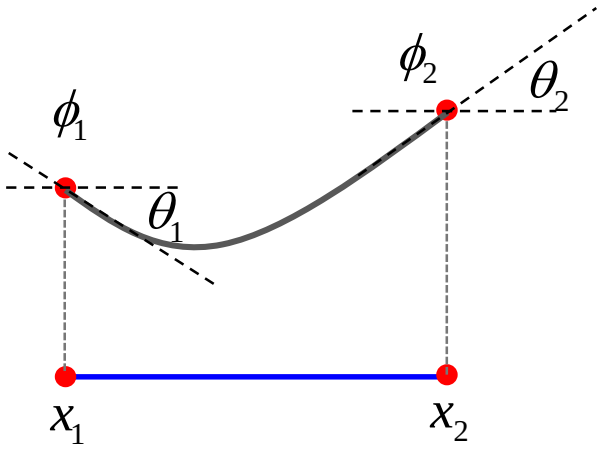
厄米单元

同时保持场函数与其导数的连续性

$$\theta_i = \left.\frac{\mathrm{d}\phi_i}{\mathrm{d}\xi}\right|_i$$



$$\phi(\xi) = \sum_{i=1}^{2} H_i^{(0)}(\xi)\phi_i + \sum_{i=1}^{2} H_i^{(1)}(\xi)\theta_i$$

$$H_i^{(0)}(\xi_j) = \delta_{ij} \qquad \left.\frac{\mathrm{d}H_i^{(0)}(\xi)}{\mathrm{d}\xi}\right|_{\xi_j} = 0$$

$$H_i^{(1)}(\xi_j) = 0 \qquad \left.\frac{\mathrm{d}H_i^{(1)}(\xi)}{\mathrm{d}\xi}\right|_{\xi_j} = \delta_{ij}$$

一般两节点梁单元！

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二维单元

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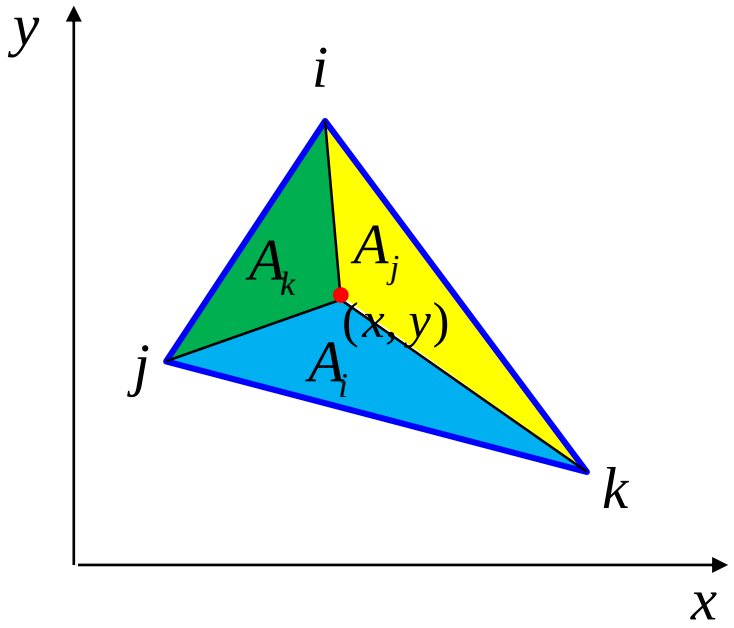
三角形的面积坐标



$$P \to P(x,y) \Leftrightarrow P(L_i, L_j, L_k)$$

$$L_i = \frac{A_i}{A}, L_j = \frac{A_j}{A}, L_k = \frac{A_k}{A}$$

$$A_i = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} \qquad A = \frac{1}{2}\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}$$

$$L_i = \frac{1}{2A}\left(a_i + b_i x + c_i y\right)$$

$$N_i = L_i, N_j = L_j, N_k = L_k$$

$$\frac{\partial L_i}{\partial x} = \frac{b_i}{2A}, \frac{\partial L_i}{\partial y} = \frac{c_i}{2A}$$

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线性三角形单元

插值函数即为面积坐标！

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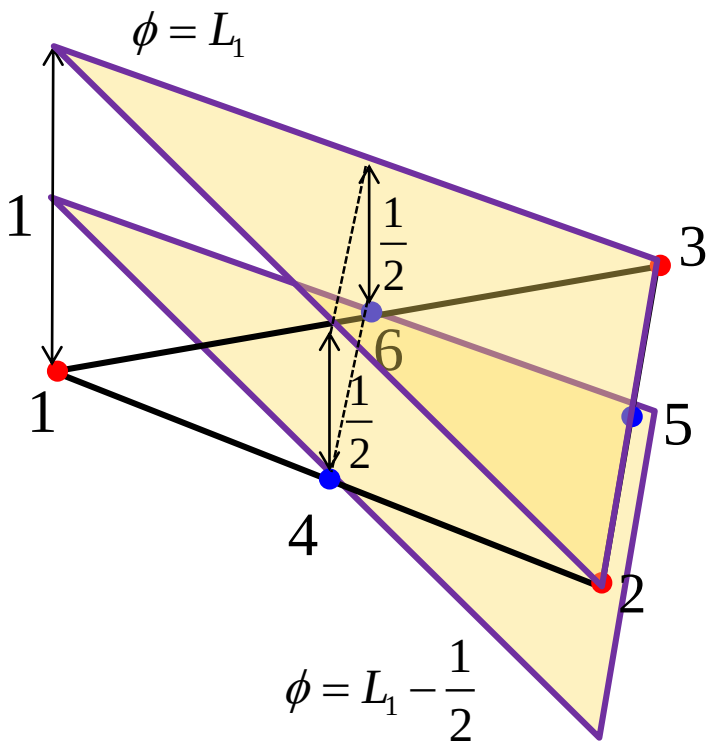
二阶三角形单元



$$N_1 = \frac{L_1 - \frac{1}{2}}{1 - \frac{1}{2}} \frac{L_1}{1} = \left(2L_1 - 1\right) L_1$$

$$N_2 = \frac{L_2 - \frac{1}{2}}{1 - \frac{1}{2}} \frac{L_2}{1} = \left(2L_2 - 1\right) L_2$$

$$N_3 = \frac{L_3 - \frac{1}{2}}{1 - \frac{1}{2}} \frac{L_3}{1} = \left(2L_3 - 1\right) L_3$$

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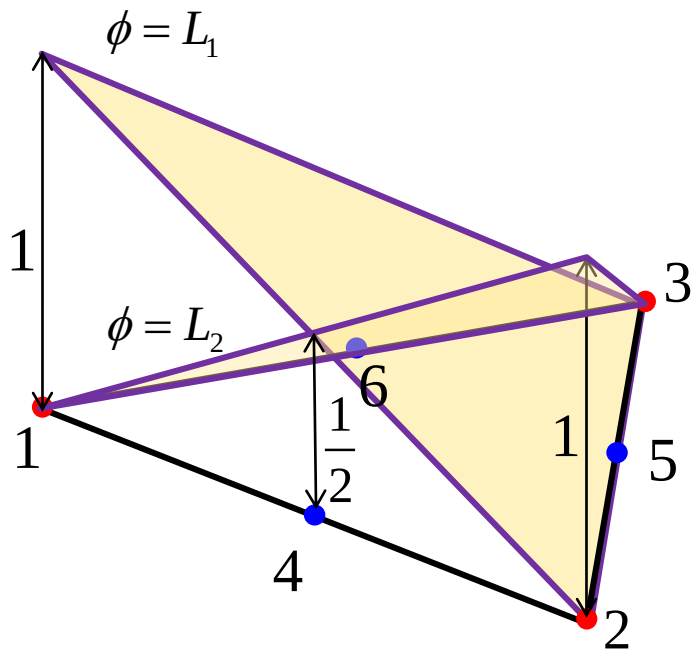
二阶三角形单元



$$N_4 = \frac{L_1}{\frac{1}{2}} \frac{L_2}{\frac{1}{2}} = 4L_1L_2$$

$$N_5 = \frac{L_2}{\frac{1}{2}} \frac{L_3}{\frac{1}{2}} = 4L_2L_3$$

$$N_6 = \frac{L_1}{\frac{1}{2}} \frac{L_2}{\frac{1}{2}} = 4L_1L_2$$

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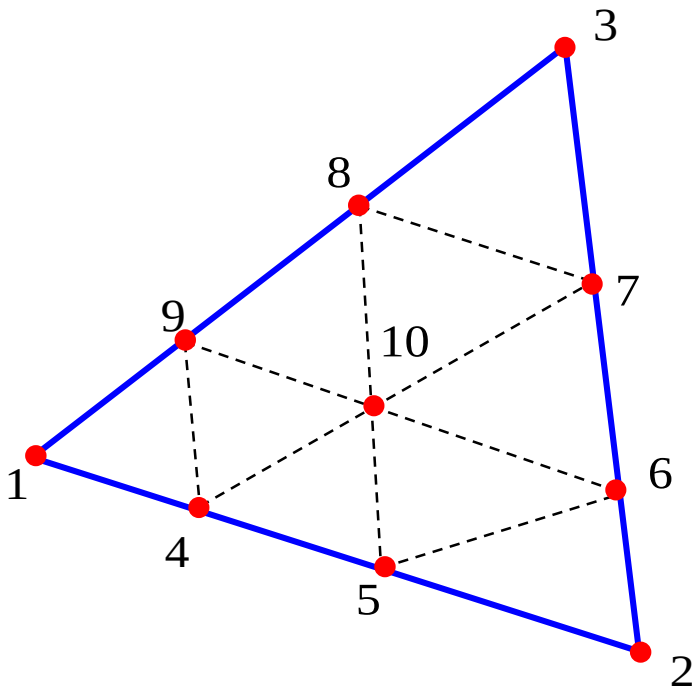
三阶三角形单元



角节点：

$$N_1=\frac{L_1-\frac{2}{3}}{1-\frac{2}{3}}\frac{L_1-\frac{1}{3}}{1-\frac{1}{3}}\frac{L_1}{1}=\frac{1}{2}\left(3L_1-1\right)\left(3L_1-2\right)L_1$$

边节点：

$$N_4=\frac{L_1}{\frac{2}{3}}\frac{L_2}{\frac{1}{3}}\frac{L_1-\frac{1}{3}}{\frac{2}{3}-\frac{1}{3}}=\frac{9}{2}L_1L_2\left(3L_1-1\right)$$

内节点：

$$N_{10}=\frac{L_1}{\frac{1}{3}}\frac{L_2}{\frac{1}{3}}\frac{L_3}{\frac{1}{3}}=27L_1L_2L_3$$

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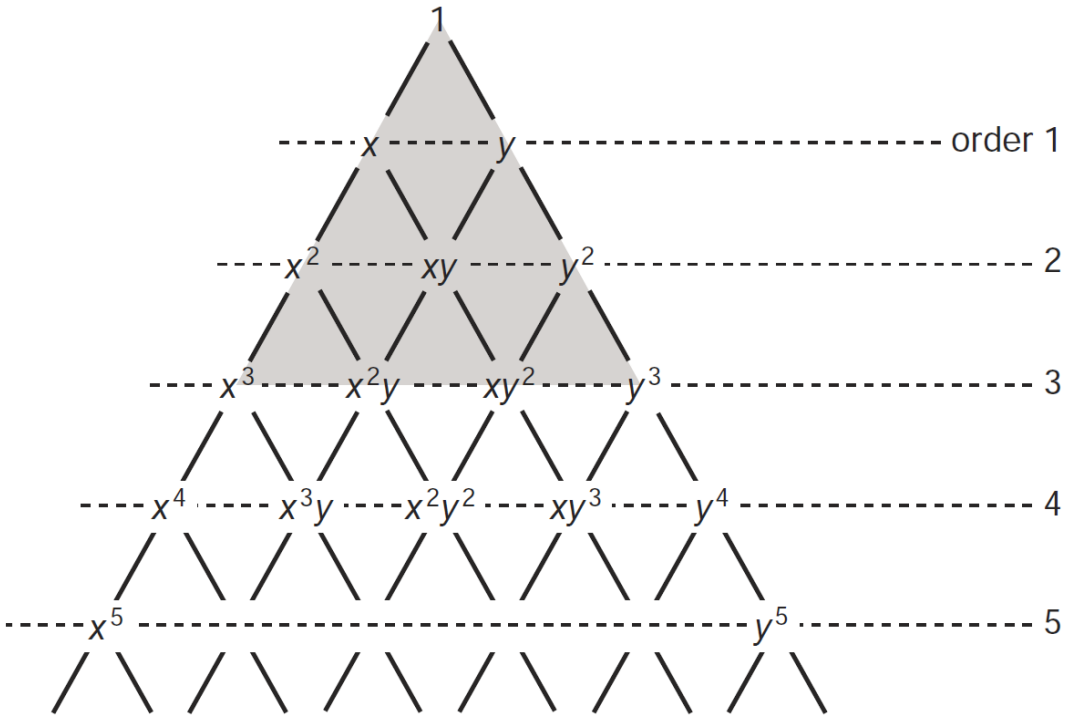
三角形单元与帕斯卡三角形



$$N_i(x) = \prod_{j=1}^{p} \frac{f_j^{(i)}(L_1, L_2, L_3)}{f_j^{(i)}(L_{1i}, L_{2i}, L_{3i})}$$

广义拉格朗日插值函数！

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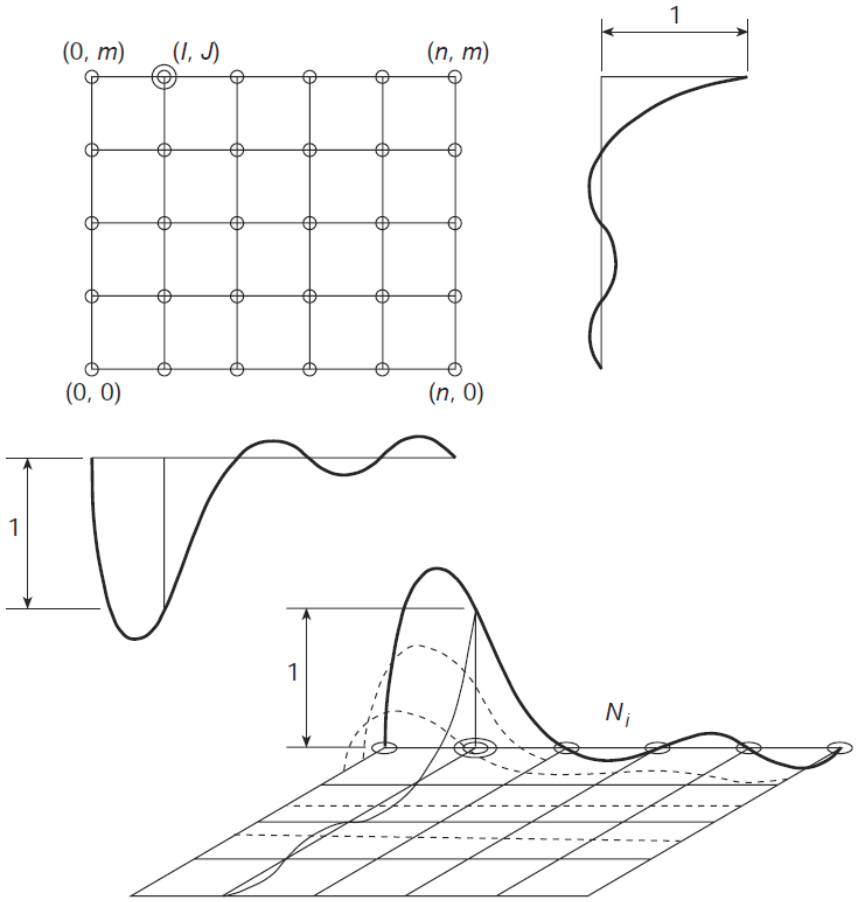
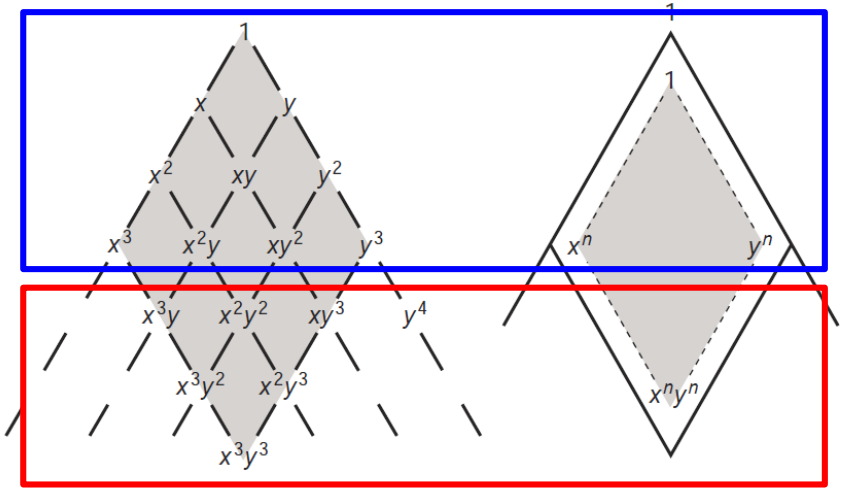
拉格朗日矩形单元



$$N_i = N_{IJ} = l_I^{(m)}(\xi) l_J^{(n)}(\eta)$$

完备多项式项



不完备多项式项

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厄米矩形单元

可用与拉格朗日矩形单元类似的方法构造！

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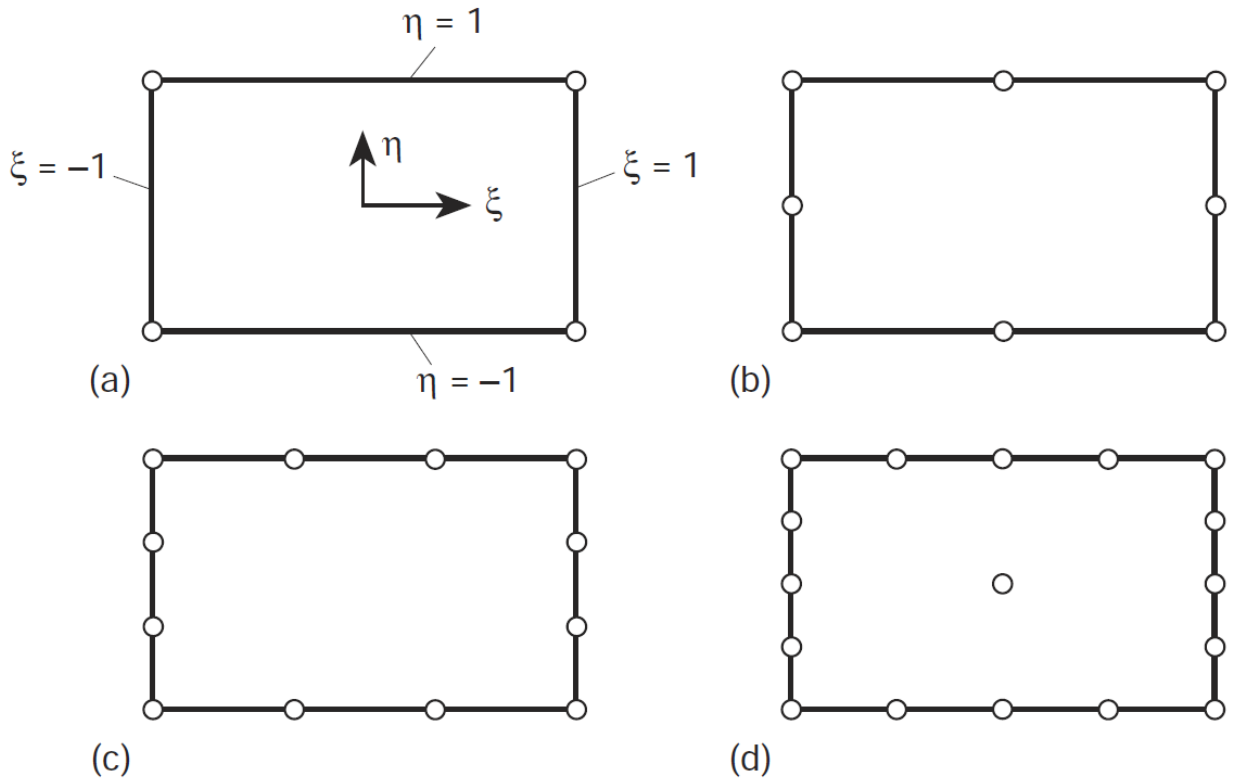
Serendipity矩形单元



移除拉格朗日单元的不完备自由度对应节点，剩余节点尽量在边上！

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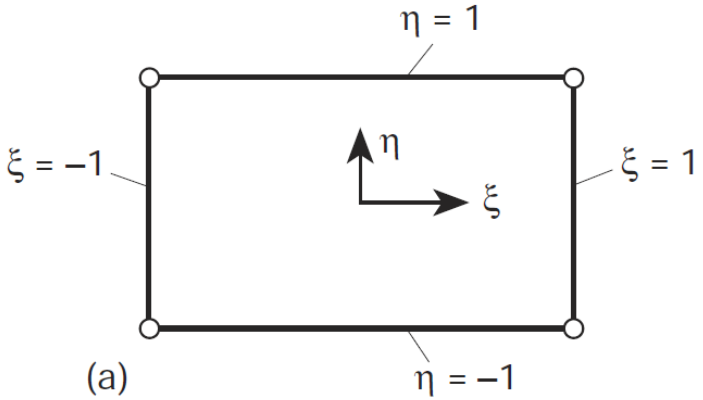
Serendipity矩形单元

递推构造方法——简单四节点单元



$$\hat{N}_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$$

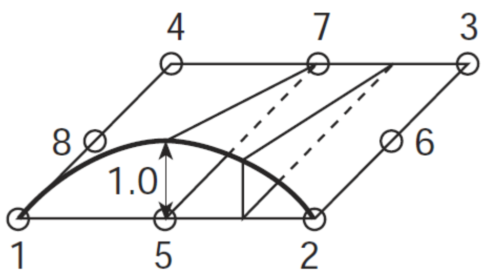
$$i = 1, 2, 3, 4$$

Serendipity矩形单元

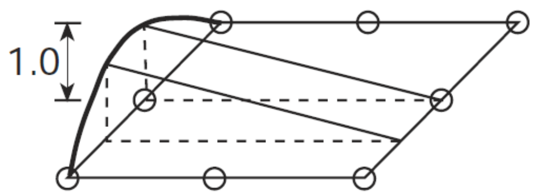
$$\hat{N}_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$$

$$i = 1, 2, 3, 4$$

递推构造方法——八节点单元

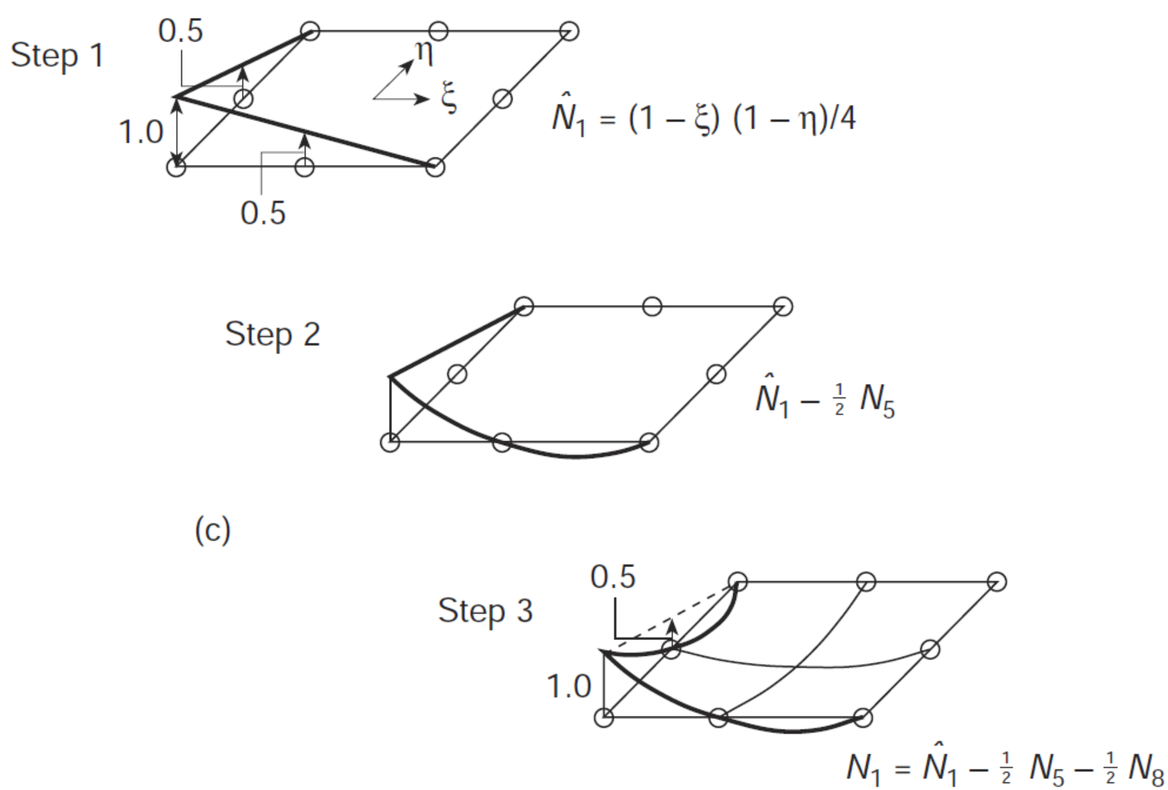


$$N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$$



$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

边节点插值函数构造



角节点插值函数修正

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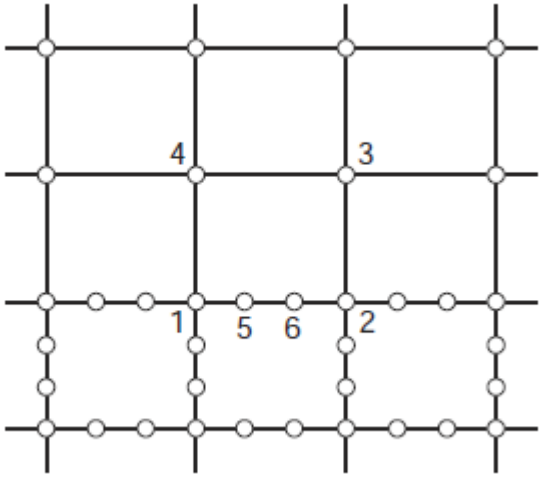
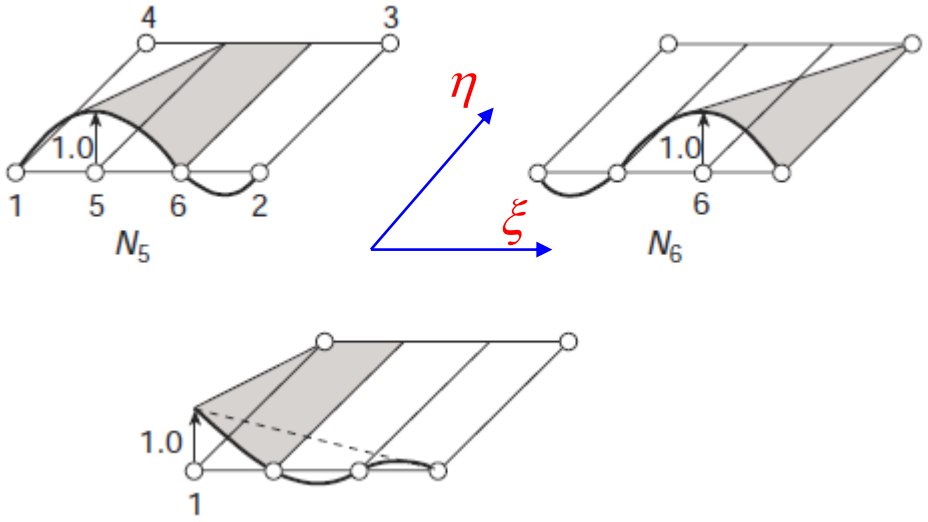
Serendipity矩形单元

递推构造方法——三次/线性过渡单元





$$N_5=\frac{1-\xi}{1+\frac{1}{3}}\frac{1+\xi}{1-\frac{1}{3}}\frac{\frac{1}{3}-\xi}{\frac{1}{3}+\frac{1}{3}}\frac{1-\eta}{1+1}=\frac{9}{32}\left(1-\xi^2\right)(1-3\xi)(1-\eta)$$

$$N_6=\frac{1-\xi}{1-\frac{1}{3}}\frac{1+\xi}{1+\frac{1}{3}}\frac{\frac{1}{3}+\xi}{\frac{1}{3}+\frac{1}{3}}\frac{1-\eta}{1+1}=\frac{9}{32}\left(1-\xi^2\right)(1+3\xi)(1-\eta)$$

$$N_1=\hat{N}_1-\frac{2}{3}N_5-\frac{1}{3}N_6$$

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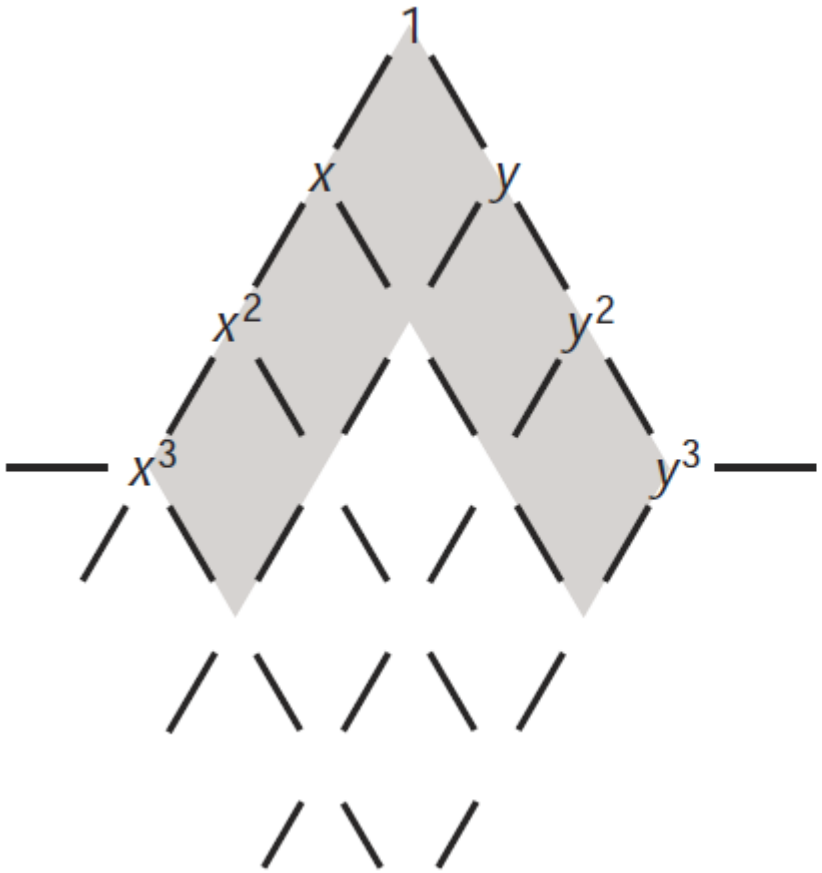
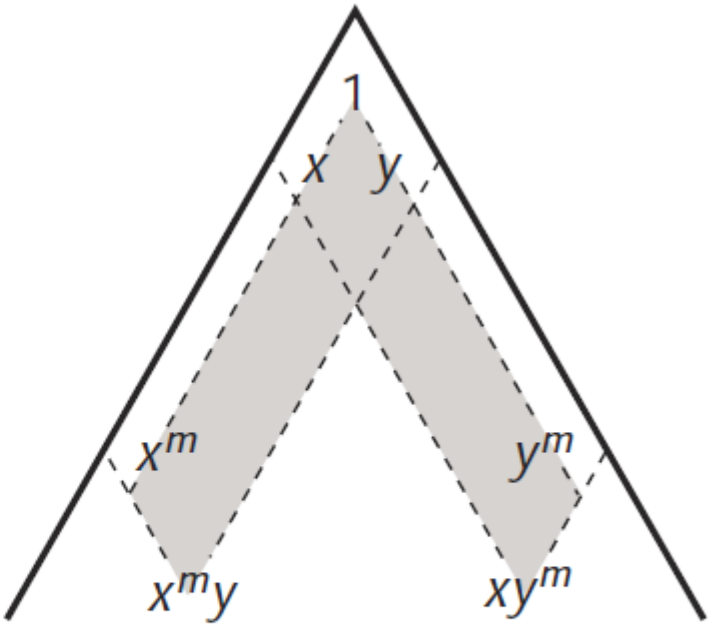
Serendipity矩形单元





完备边界型Serendipity单元插值函数幂次

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大作业一（须提交分析报告）

对于函数：$\phi(x,y)=\sin x\cos y,\quad (x,y)\in[0,\pi/2]\otimes[0,\pi/2]$

完成以下计算。

1. 采用3节点、6节点三角形单元，4节点、8节点矩形单元，对函数进行插值，画出原函数-插值函数、原函数导数-插值导数的图形，并讨论计算结果。（插值网格取1×1，2×2，5×5，三角形单元在矩形单元中采用对角线斜跨划分即可。）
2. 计算原函数插值和插值导数的2-范数误差，将误差-单元长度（E-h）曲线绘制在双对数坐标系中，观察和讨论计算所得曲线的性质。（至少变化5个单元长度，跨过3个长度量级，才可能得到稳定且有规律的结果。）

$$\left\|f-\hat{f}\right\|_2=\left[\int_\Omega\left(f-\hat{f}\right)^2\mathrm{d}\Omega\right]^{\frac{1}{2}}$$

4月20日提交！

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今天就到这里，

明天的事儿明天再说！

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