有限单元法研究生核心课程

第十二讲 非线性问题概述

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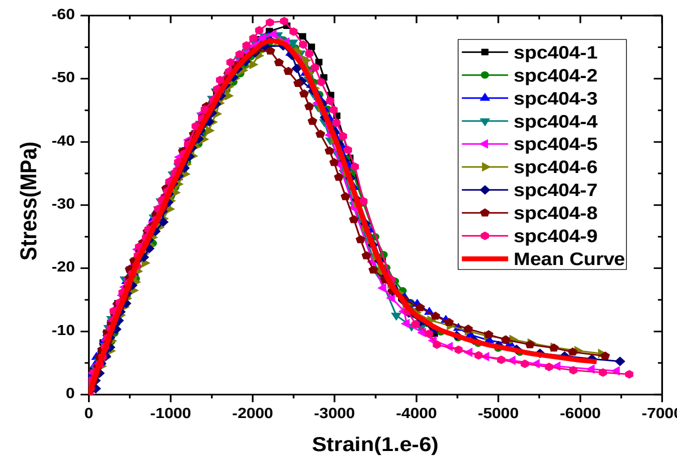
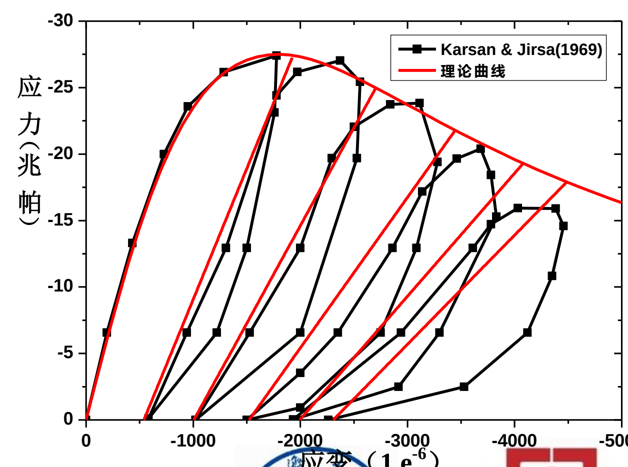
非线性问题——材料非线性





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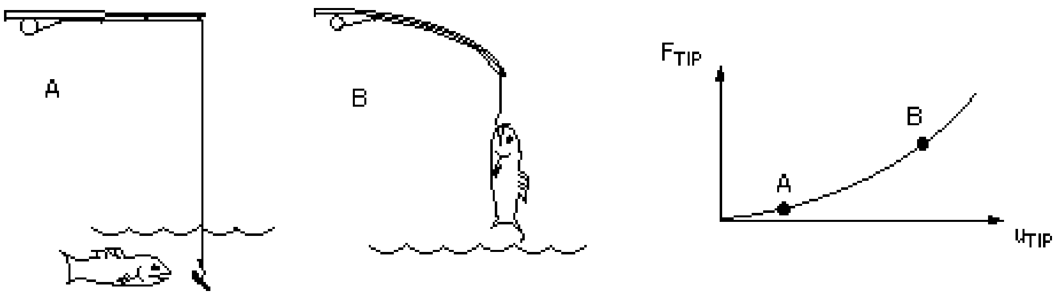
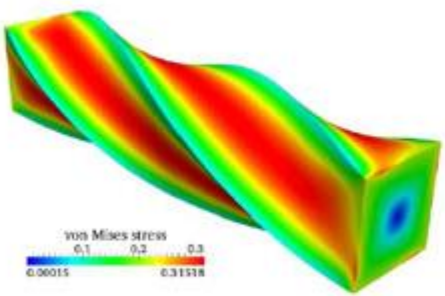
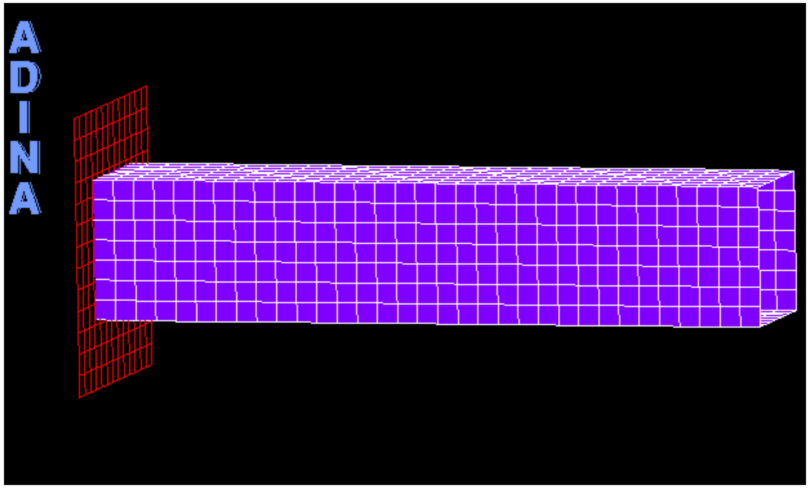
非线性问题——几何非线性







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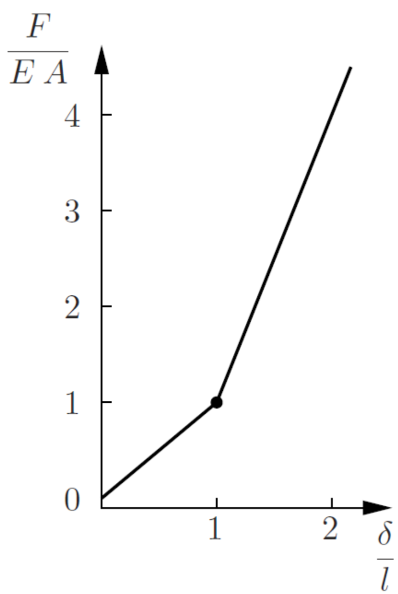
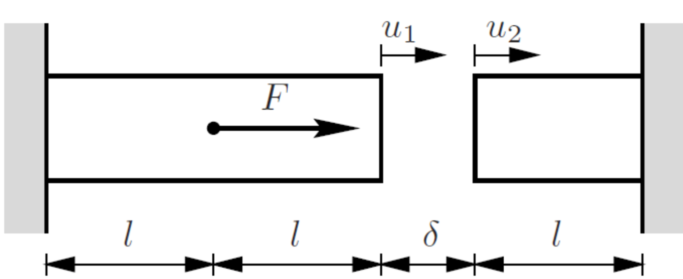
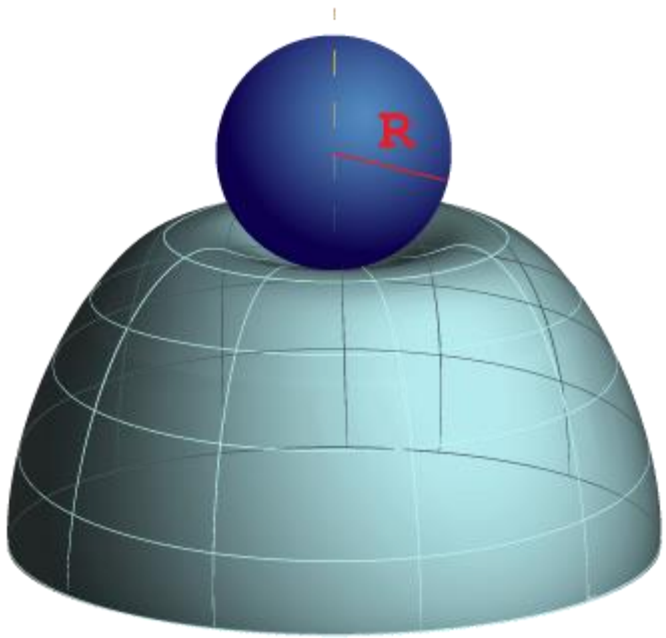
非线性问题——接触（约束）非线性







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虚功原理（固体控制方程弱形式）

$$\int_{\Omega}\delta\varepsilon_{ij}\sigma_{ij}\mathrm{d}\Omega=\int_{\Gamma_s}\delta u_i t_i\mathrm{d}\Gamma+\int_{\Omega}\delta u_i b_i\mathrm{d}\Omega$$

s.t. $C_j(u_i,\varepsilon_{ij},\sigma_{ij}) = / \le / < 0$ in/on □

接触（约束）非线性

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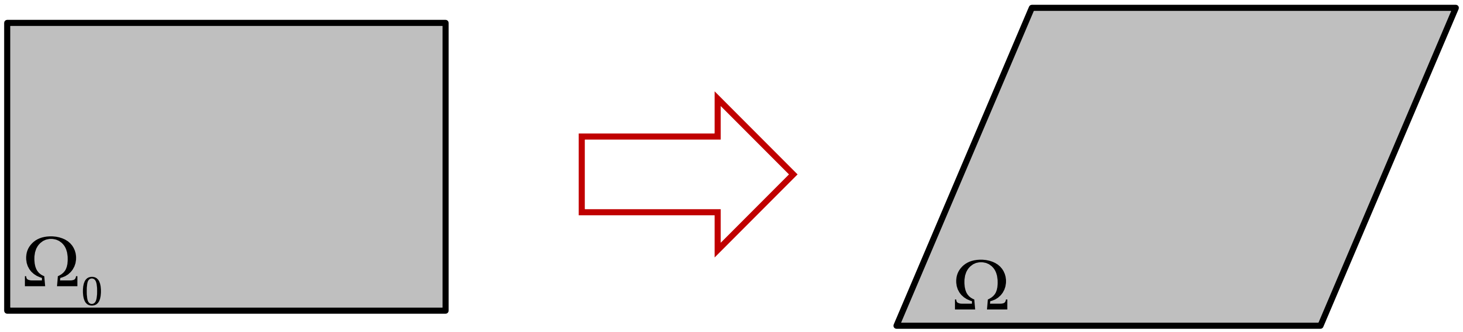
几何非线性的引入

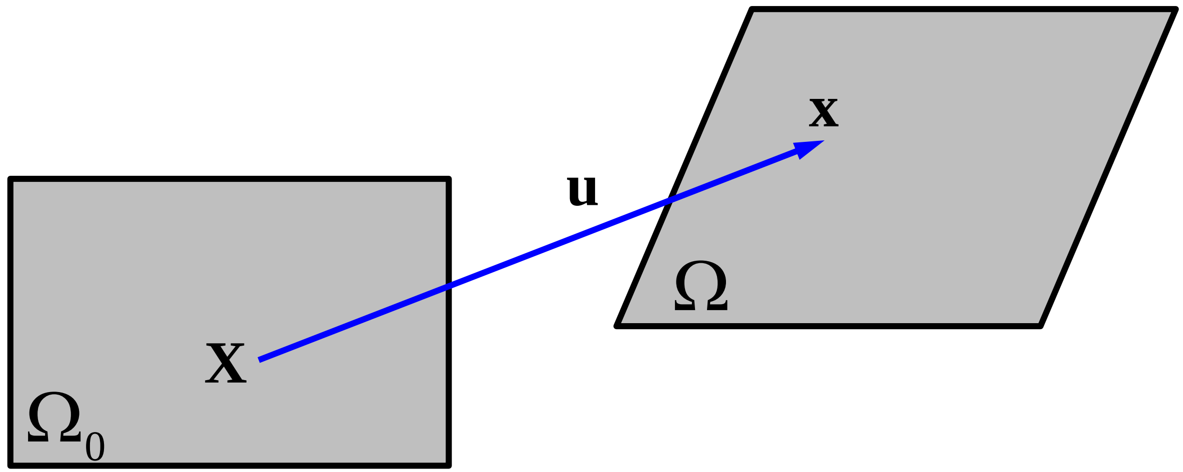


变形前　　　　变形后

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0$$

$$\int_{\Omega} \delta \varepsilon_{ij} \sigma_{ij} \mathrm{d}\Omega = \int_{\Gamma_s} \delta u_i t_i \mathrm{d}\Gamma + \int_{\Omega} \delta u_i b_i \mathrm{d}\Omega$$

几何非线性的引入

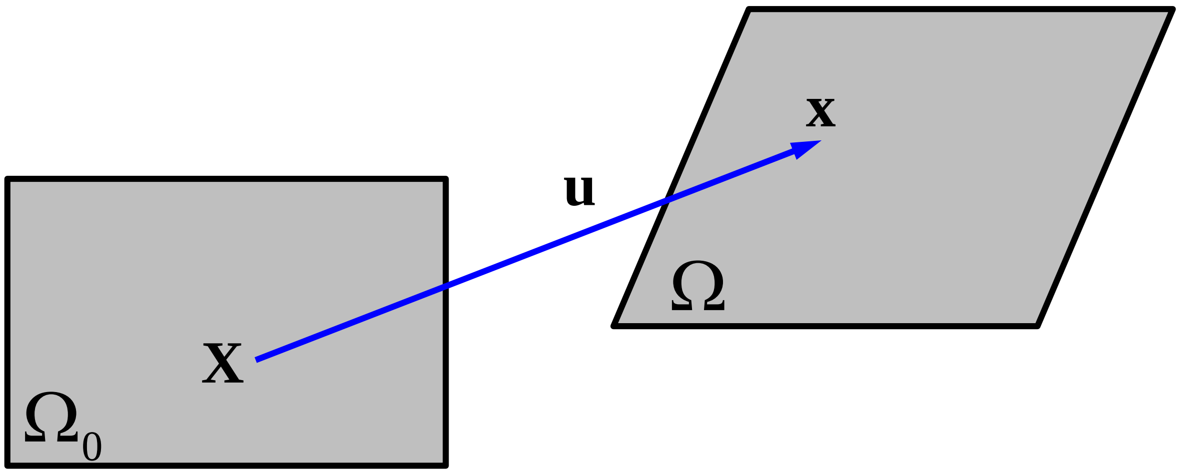


位移： $\mathbf{u}=\mathbf{x}-\mathbf{X}$

$$\mathbf{u}(\mathbf{x})=\mathbf{x}-\mathbf{X}=\mathbf{x}-\phi(\mathbf{x})$$

$$\mathbf{u}(\mathbf{X})=\mathbf{x}-\mathbf{X}=\varphi(\mathbf{X})-\mathbf{X}$$

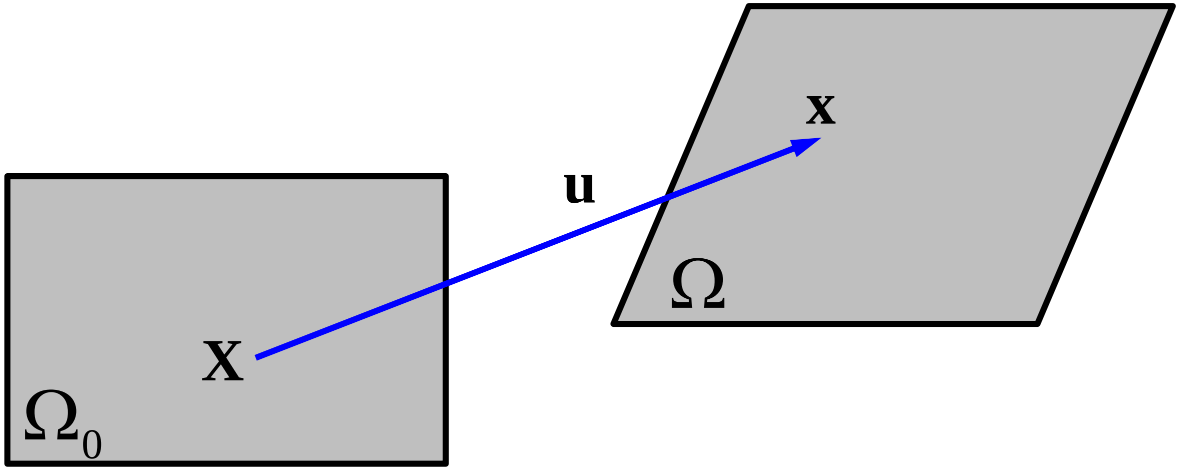
几何非线性的引入



应变： ε_{ij} 位移对坐标的偏导数。。。

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

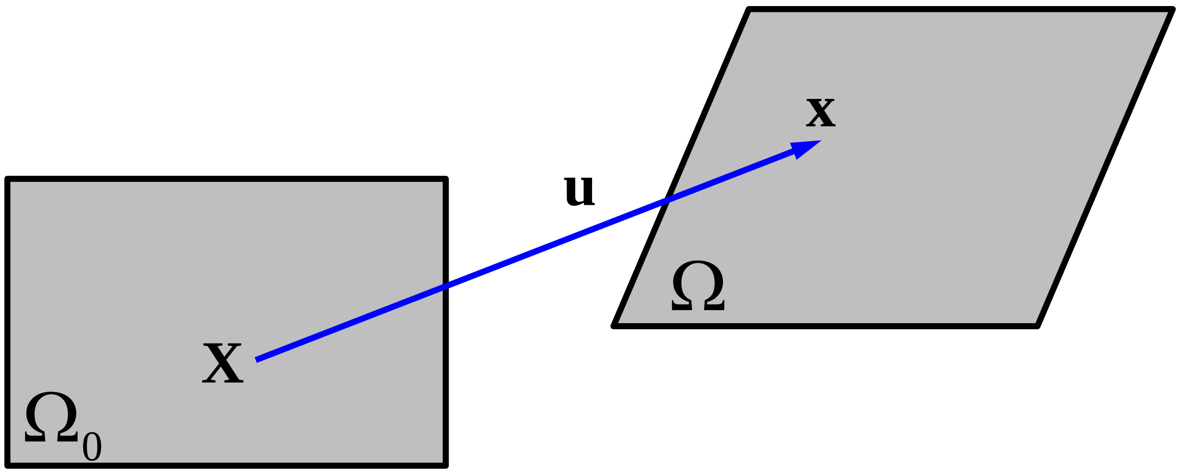
几何非线性的引入



应力： σ_{ij} Cauchy应力定律：

$$t_j = \sigma_{ij} n_i$$

几何非线性的引入

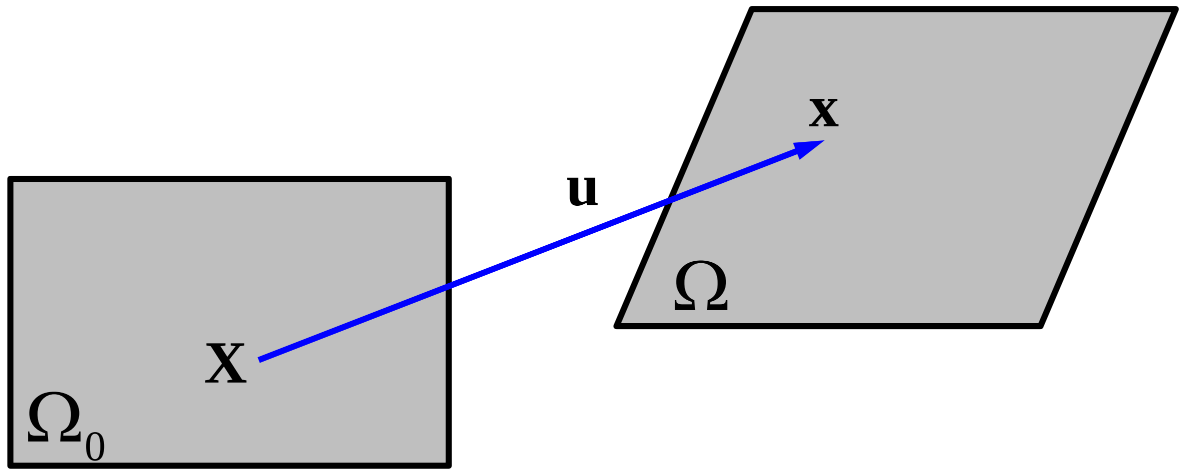


$$\Omega = \Omega(\mathbf{x})$$

体积和面积：

$$\Gamma = \Gamma(\mathbf{x})$$

几何非线性的引入



体力和面力： t_i b_i

由于实际问题的不同，既可能是位移的函数，也有可能不是。

几何非线性的引入

$$\int_{\Omega}\delta\varepsilon_{ij}\sigma_{ij}\mathrm{d}\Omega=\int_{\Gamma_s}\delta u_i t_i\mathrm{d}\Gamma+\int_{\Omega}\delta u_i b_i\mathrm{d}\Omega$$

控制方程中涉及到的几何度量均与未知位移有关！

直接在未知位型上求解！	映射到某个已知位型上求解！
Euler 格式	Lagrange 格式
流体	固体

几何非线性的引入

$$\int_{\Omega} \delta\varepsilon_{ij}\sigma_{ij}\mathrm{d}\Omega = \int_{\Gamma_s} \delta u_i t_i \mathrm{d}\Gamma + \int_{\Omega} \delta u_i b_i \mathrm{d}\Omega$$

梯度变换：$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial X_j}\frac{\partial X_j}{\partial x_i}$ $\quad J = \frac{\partial x_j}{\partial X_i} = \left(\frac{\partial X_j}{\partial x_i}\right)^{-1}$

Jacobian

积分变换：$\int_{\Omega} \bullet \mathrm{d}\Omega = \int_{\Omega_0} \bullet \left|J\right| \mathrm{d}\Omega_0$

材料非线性

全量形式： $\sigma_{ij} = f_{ij}\left(\varepsilon_{kl}\right)$

微分形式： $\dot{\sigma}_{ij} = C_{ijkl}^{\mathrm{T}} \dot{\varepsilon}_{kl}$

增量形式： $\Delta\sigma_{ij} = C_{ijkl}^{\mathrm{T}} \Delta\varepsilon_{kl}$

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非线性方程

$$\int_{\Omega} \delta\varepsilon_{ij}\sigma_{ij}\,\mathrm{d}\Omega = \int_{\Gamma_s} \delta u_i t_i\,\mathrm{d}\Gamma + \int_{\Omega} \delta u_i b_i\,\mathrm{d}\Omega$$

$$\mathbf{f}^{\mathrm{int}}(\mathbf{u}) = \mathbf{f}^{\mathrm{ext}}$$

非线性方程求解——NR方法

$$\mathbf{f}(\mathbf{u}+\Delta\mathbf{u})=\mathbf{p}+\Delta\mathbf{p} \qquad \mathbf{f}(\mathbf{u}+\Delta\mathbf{u}_n)+\left.\frac{\partial\mathbf{f}(\mathbf{u})}{\partial\mathbf{u}}\right|_{\mathbf{u}+\Delta\mathbf{u}_n}\delta\mathbf{u}_n\approx\mathbf{p}+\Delta\mathbf{p}$$

$$\begin{cases}\delta\mathbf{u}_n=\mathbf{K}_\mathrm{T}^{-1}\ \mathbf{u}+\Delta\mathbf{u}_n\ \ \mathbf{p}+\Delta\mathbf{p}-\mathbf{f}(\mathbf{u}+\Delta\mathbf{u}_n)\\ \Delta\mathbf{u}_{n+1}=\Delta\mathbf{u}_n+\delta\mathbf{u}_n\end{cases}$$

$$\mathbf{K}_\mathrm{T}\ \ \mathbf{u}+\Delta\mathbf{u}_n\ =\left.\frac{\partial\mathbf{f}(\mathbf{u})}{\partial\mathbf{u}}\right|_{\mathbf{u}+\Delta\mathbf{u}_n}$$ 切线刚度矩阵

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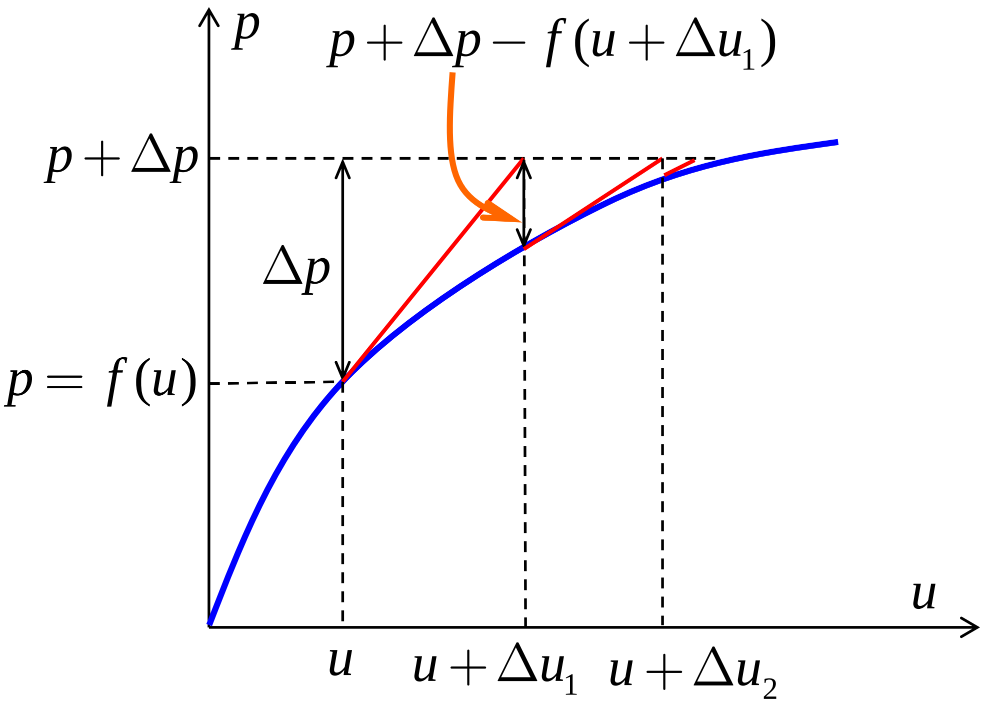
非线性方程求解——NR方法



$$\begin{cases} \delta \mathbf{u}_n = \left(\left. \dfrac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}} \right|_{\mathbf{u}+\Delta \mathbf{u}_n} \right)^{-1} \mathbf{p}+\Delta \mathbf{p}-\mathbf{f}(\mathbf{u}+\Delta \mathbf{u}_n) \\ \Delta \mathbf{u}_{n+1} = \Delta \mathbf{u}_n + \delta \mathbf{u}_n \end{cases}$$

NEWTON-RAPHSON方法若干评论

- 二阶精度（quadratic convergence rate）
- 收敛很快（3~5步迭代）且基本不依赖于问题的维度
- 局部收敛——增量法
- 刚度矩阵计算量较大
- 对于某些问题，其刚度矩阵奇异，进而导致计算困难或者失败
- 对于一般问题，一般首先尝试采用N-R方法，该方法计算失败，再考虑其它方法

QN方法

$$\begin{cases}\delta\mathbf{u}_n = \mathbf{K}_S^{-1}(\mathbf{u}+\Delta\mathbf{u}_n)\left(\mathbf{p}+\Delta\mathbf{p}-\mathbf{f}(\mathbf{u}+\Delta\mathbf{u}_n)\right)\\ \Delta\mathbf{u}_{n+1} = \Delta\mathbf{u}_n + \delta\mathbf{u}_n\end{cases}$$

$\mathbf{K}_S(\mathbf{u}+\Delta\mathbf{u}_n)$ 割线刚度矩阵

$$\begin{cases}\mathbf{f}(\mathbf{u}) = \mathbf{p}\\ \mathbf{f}(\mathbf{u}_{\mathrm{ref}}) = \mathbf{p}_{\mathrm{ref}}\end{cases} \Rightarrow \mathbf{f}(\mathbf{u})-\mathbf{f}(\mathbf{u}_{\mathrm{ref}}) = \mathbf{p}-\mathbf{p}_{\mathrm{ref}} \Rightarrow \mathbf{K}_S(\mathbf{u}-\mathbf{u}_{\mathrm{ref}}) = \mathbf{p}-\mathbf{p}_{\mathrm{ref}}$$

多维情况下满足上述关系
的割线刚度矩阵不唯一

BROYDEN UPDATE

$$\mathbf{K}_S \Delta \mathbf{u} = \Delta \mathbf{p}$$

$$\mathbf{K}_S = \mathbf{K}_0 + \frac{(\Delta \mathbf{p} - \mathbf{K}_0 \Delta \mathbf{u}) \Delta \mathbf{u}^T}{\Delta \mathbf{u}^T \Delta \mathbf{u}}$$

Good Broyden

$$\mathbf{K}_S^{-1} = \mathbf{K}_0^{-1} + \frac{(\Delta \mathbf{u} - \mathbf{K}_0^{-1} \Delta \mathbf{p}) \Delta \mathbf{p}^T}{\Delta \mathbf{p}^T \Delta \mathbf{p}}$$

Bad Broyden

BFGS UPDATE

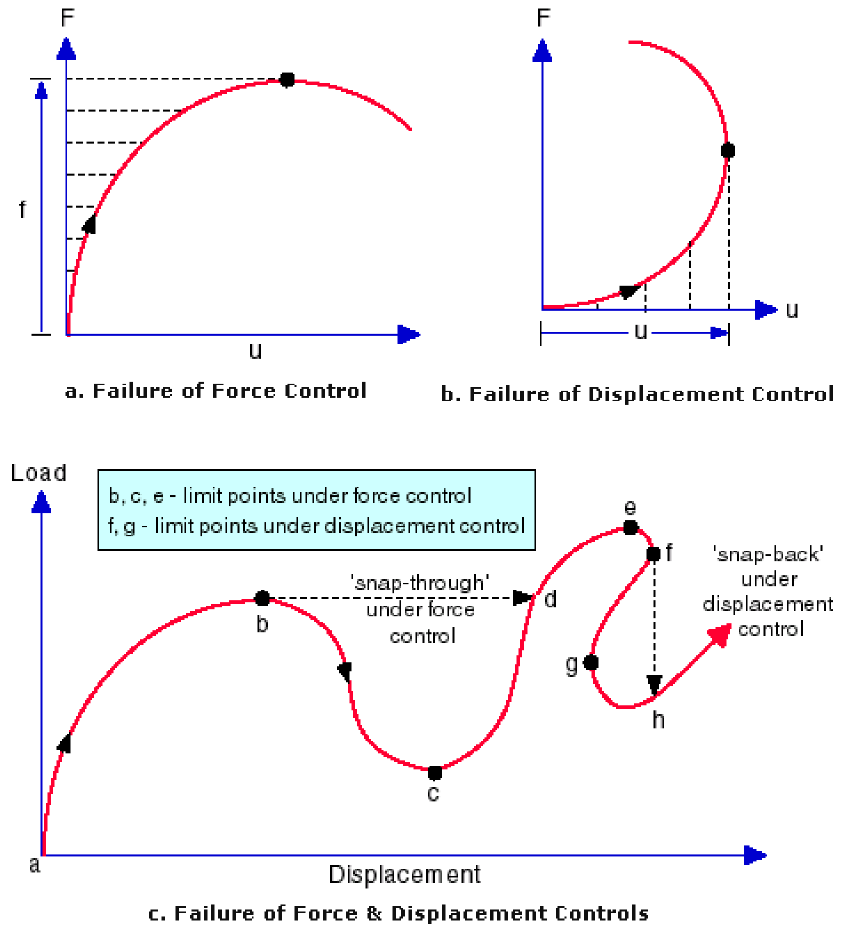
$$\mathbf{K}_S \Delta \mathbf{u} = \Delta \mathbf{p}$$

$$\mathbf{K}_S = \mathbf{K}_0 + \frac{\Delta \mathbf{p} \Delta \mathbf{p}^{\mathrm{T}}}{\Delta \mathbf{p}^{\mathrm{T}} \Delta \mathbf{u}} - \frac{\mathbf{K}_0 \Delta \mathbf{u} \Delta \mathbf{u}^{\mathrm{T}} \mathbf{K}_0}{\Delta \mathbf{u}^{\mathrm{T}} \mathbf{K}_0 \Delta \mathbf{u}}$$

$$\mathbf{K}_S^{-1} = \mathbf{K}_0^{-1} + \frac{\Delta \mathbf{u} \Delta \mathbf{u}^{\mathrm{T}}}{\Delta \mathbf{u}^{\mathrm{T}} \Delta \mathbf{p}} - \frac{\mathbf{K}_0^{-1} \Delta \mathbf{p} \Delta \mathbf{p}^{\mathrm{T}} \mathbf{K}_0^{-1}}{\Delta \mathbf{p}^{\mathrm{T}} \mathbf{K}_0^{-1} \Delta \mathbf{p}}$$

对称、正定！

复杂非线性问题



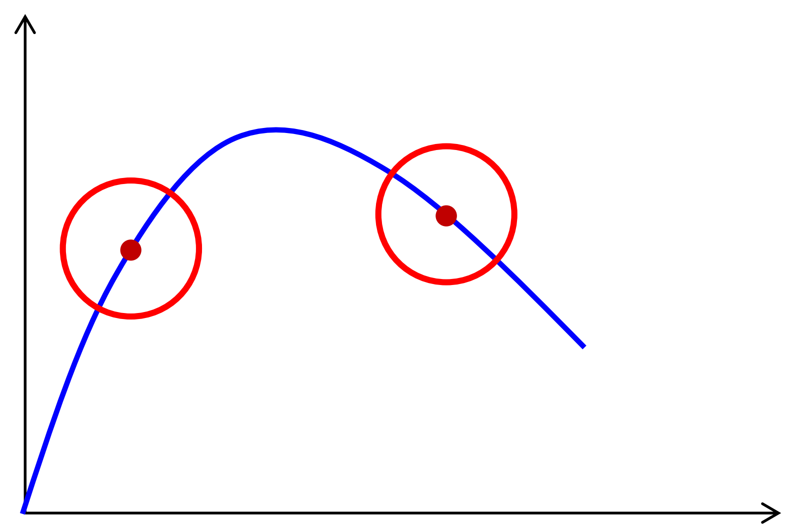
Karman T, Tsien H-S. The buckling of thin cylindrical shells under axial compression. J Aeronaut Sci 1941;8:303–12.

弧长（ARCH-LENGTH）方法

$$\mathbf{f}(\mathbf{u},\lambda)=\mathbf{g}(\mathbf{u})-\lambda_i\mathbf{q}$$

$$\mathbf{f}(\mathbf{u}+\Delta\mathbf{u},\lambda+\Delta\lambda)=\mathbf{g}(\mathbf{u}+\Delta\mathbf{u})-\ \lambda_i+\Delta\lambda\ \ \mathbf{q}$$

$$a=\Delta\mathbf{u}^T\Delta\mathbf{u}+\Delta\lambda^2\psi^2\mathbf{q}^T\mathbf{q}-\Delta l^2=0$$



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弧长（ARCH-LENGTH）方法

$$\begin{cases} \mathbf{f}(\mathbf{u}+\Delta\mathbf{u},\lambda+\Delta\lambda)=\mathbf{g}(\mathbf{u}+\Delta\mathbf{u})-(\lambda_i+\Delta\lambda)\mathbf{q} \\ a=\Delta\mathbf{u}^T\Delta\mathbf{u}+\Delta\lambda^2\psi^2\mathbf{q}^T\mathbf{q}-\Delta l^2=0 \end{cases}$$

用N-R方法求解上述方程

$$\begin{bmatrix} \left.\dfrac{\partial\mathbf{f}(\mathbf{u},\lambda+\Delta\lambda)}{\partial\mathbf{u}}\right|_{\mathbf{u}+\Delta\mathbf{u}_n} & -\mathbf{q} \\ 2\Delta\mathbf{u}_n^T & 2\Delta\lambda_n\psi^2\mathbf{q}^T\mathbf{q} \end{bmatrix} \begin{bmatrix} \delta\Delta\mathbf{u}_n \\ \delta\Delta\lambda_n \end{bmatrix}$$

$$=\begin{bmatrix} \mathbf{g}(\mathbf{u}+\Delta\mathbf{u}_n)-(\lambda_i+\Delta\lambda_n)\mathbf{q}-\mathbf{f}(\mathbf{u}+\Delta\mathbf{u}_n,\lambda+\Delta\lambda_n) \\ \Delta l^2-(\Delta\mathbf{u}_n^T\Delta\mathbf{u}_n+\Delta\lambda_n^2\psi^2\mathbf{q}^T\mathbf{q}) \end{bmatrix}$$

$$\begin{bmatrix} \Delta\mathbf{u}_{n+1} \\ \Delta\lambda_{n+1} \end{bmatrix}=\begin{bmatrix} \Delta\mathbf{u}_n \\ \Delta\lambda_n \end{bmatrix}+\begin{bmatrix} \delta\Delta\mathbf{u}_n \\ \delta\Delta\lambda_n \end{bmatrix}$$

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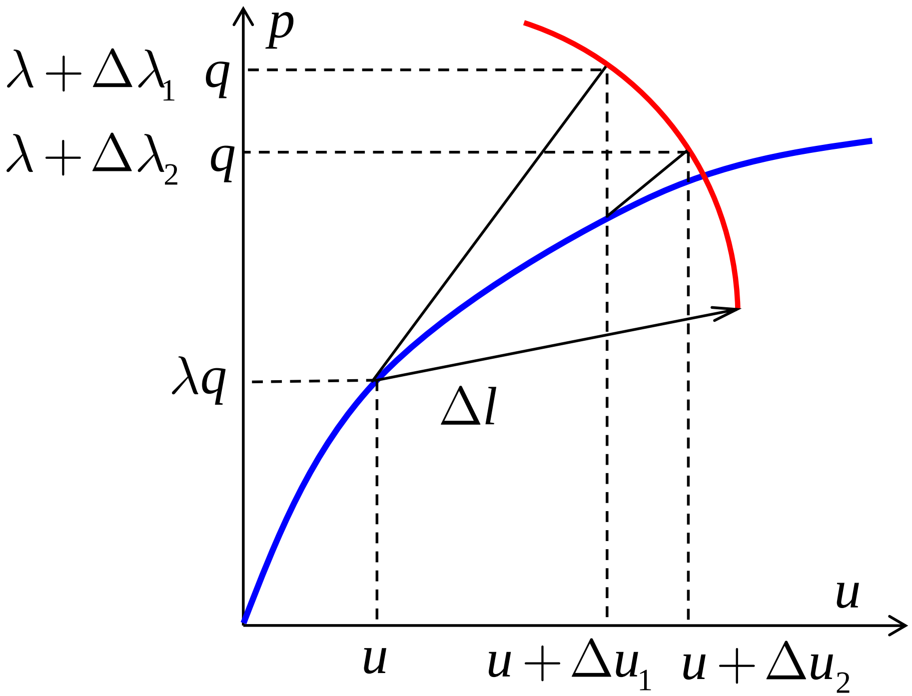
弧长（ARCH-LENGTH）方法



$$\begin{cases} \mathbf{K}\delta\Delta\mathbf{u}_n = \mathbf{g}(\mathbf{u} + \Delta\mathbf{u}_n) - \delta\Delta\lambda_n\mathbf{q} \\ \Delta\mathbf{u}_{n+1} = \Delta\mathbf{u}_n + \delta\Delta\mathbf{u}_n \end{cases}$$

$$\Delta\mathbf{u}^T\Delta\mathbf{u} + \Delta\lambda^2\psi^2\mathbf{q}^T\mathbf{q} - \Delta l^2 = 0$$

$$c_1\delta\Delta\lambda_n^2 + c_2\delta\Delta\lambda_n + c_3 = 0$$

Crisfield, M.A., 1981. A fast incremental/iterative solution procedure that handles snap-through. *Computer and Structures*, **13**:55-62.

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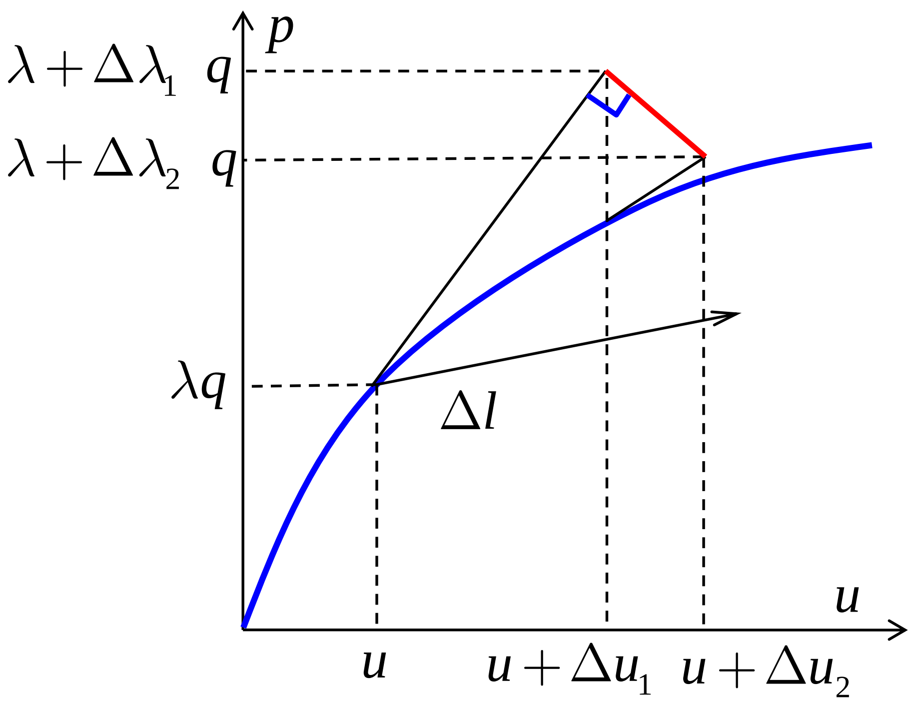
弧长（ARCH-LENGTH）方法



Riks, E., 1979. An incremental approach to the solution of snapping and buckling problems. *International Journal of Solids and Structures*, **15**:529-551.

显式方法（EXPLICIT）

不用刚度矩阵！？！？

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今天就到这里，

明天的事儿明天再说！

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