有限单元法研究生核心课程

第十一讲 平板壳单元（下）

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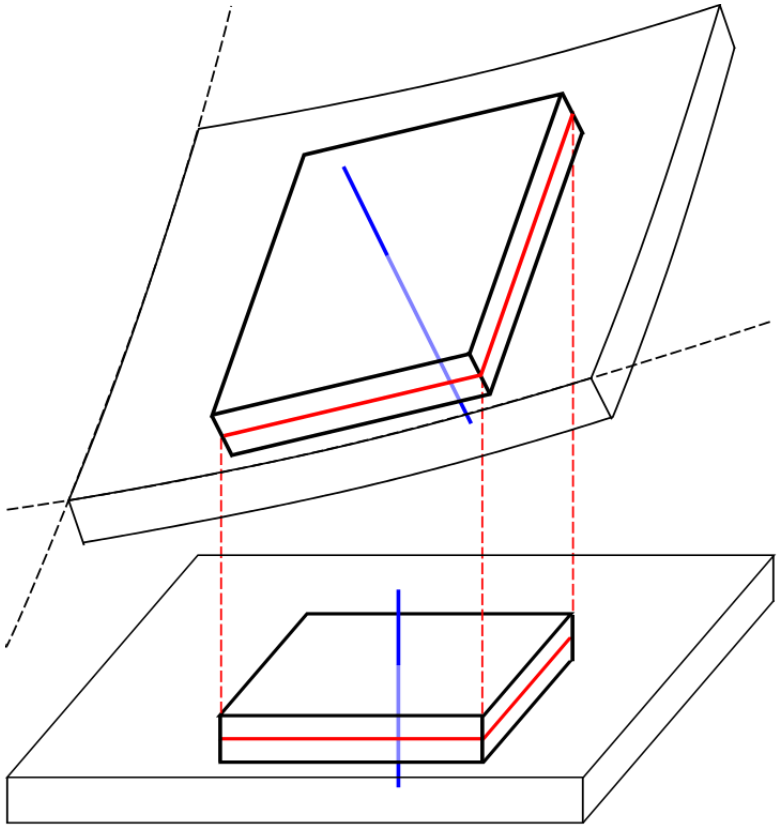
厚板理论

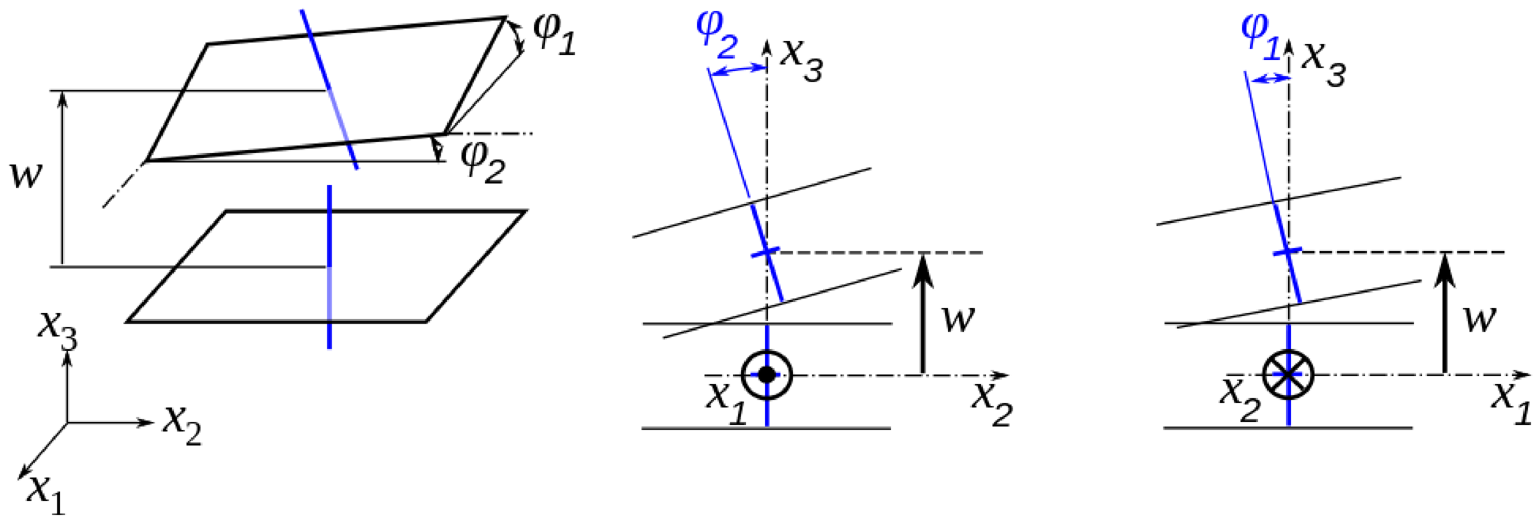
Mindlin plate theory

KIRCHHOFF假定

- 垂直与板中性面的直线变形后仍保持直线。
- 垂直与板中性面的直线变形后仍垂直与中性面。
- 板的厚度变形前后保持不变。



厚板的位移



$$u_3 = w$$

$$u_1 = u_1^0 - x_3\varphi_1 = u_1^0 - x_3\frac{\partial w}{\partial x_1} = u_1^0 - x_3 w_{,1}$$

$$u_2 = u_2^0 - x_3\varphi_2 = u_2^0 - x_3\frac{\partial w}{\partial x_2} = u_2^0 - x_3 w_{,2}$$

厚板的应变

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1}=\frac{\partial u_1^0}{\partial x_1}-x_3\frac{\partial \varphi_1}{\partial x_1}\qquad \varepsilon_{22}=\frac{\partial u_2}{\partial x_2}=\frac{\partial u_2^0}{\partial x_2}-x_3\frac{\partial \varphi_2}{\partial x_2}$$

$$\varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)=\frac{1}{2}\left(\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}\right)-x_3\frac{1}{2}\left(\frac{\partial \varphi_2}{\partial x_1}+\frac{\partial \varphi_1}{\partial x_2}\right)$$

$$\varepsilon_{31}=\frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)=\frac{1}{2}\left(w_{,1}-\varphi_1\right)$$

$$\varepsilon_{33}=\frac{\partial u_3}{\partial x_3}=w_{,3}=0$$

$$\varepsilon_{32}=\frac{1}{2}\left(\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}\right)=\frac{1}{2}\left(w_{,2}-\varphi_2\right)$$

厚板的应变

$$\varepsilon_{ij}^{0}=\frac{1}{2}\left(\frac{\partial u_i^0}{\partial x_j}+\frac{\partial u_j^0}{\partial x_i}\right)\qquad i,j=1 \text{ or } 2$$

$$\varepsilon_{ij}=\varepsilon_{ij}^{0}-x_3\frac{1}{2}\left(\frac{\partial \varphi_i}{\partial x_j}+\frac{\partial \varphi_j}{\partial x_i}\right)\quad i,j=1 \text{ or } 2$$

$$\varepsilon_{3i}=\frac{k_s}{2}\left(w_{,i}-\varphi_i\right)\quad i=1,2$$

k_s 剪切修正因子

$$\varepsilon_{33}=0$$

厚板应力和内力

厚板应力：

$\sigma_{ij} \quad i, j = 1 \text{ or } 2$ 与应变分量对应

σ_{13}, σ_{23} 截面剪应力 σ_{33} 板表面压力

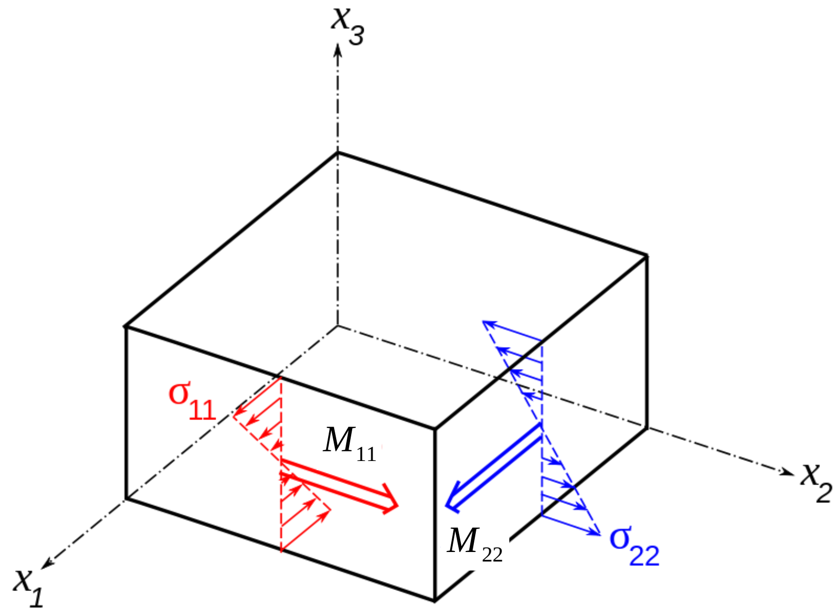
厚板内力：

$$N_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} dx_3$$ 薄膜力

$$M_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} x_3 dx_3$$ 弯矩、扭矩

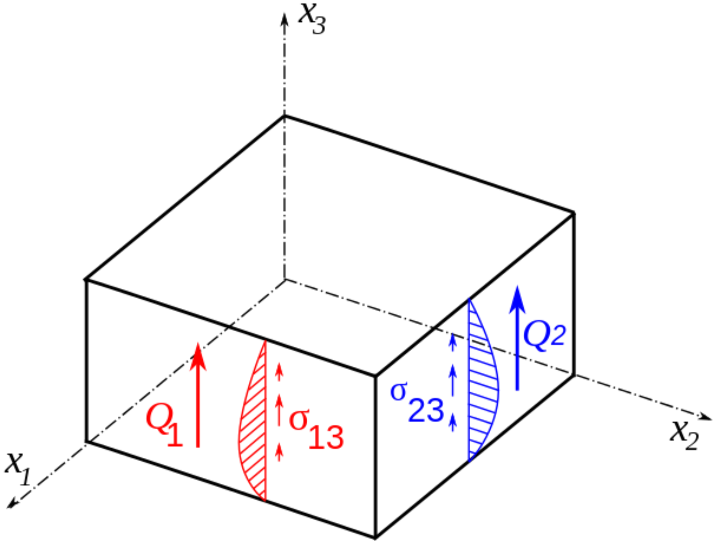
$$Q_{13} = \int_{-h/2}^{h/2} \sigma_{13} dx_3 \quad Q_{23} = \int_{-h/2}^{h/2} \sigma_{23} dx_3$$ 剪力

板的内力



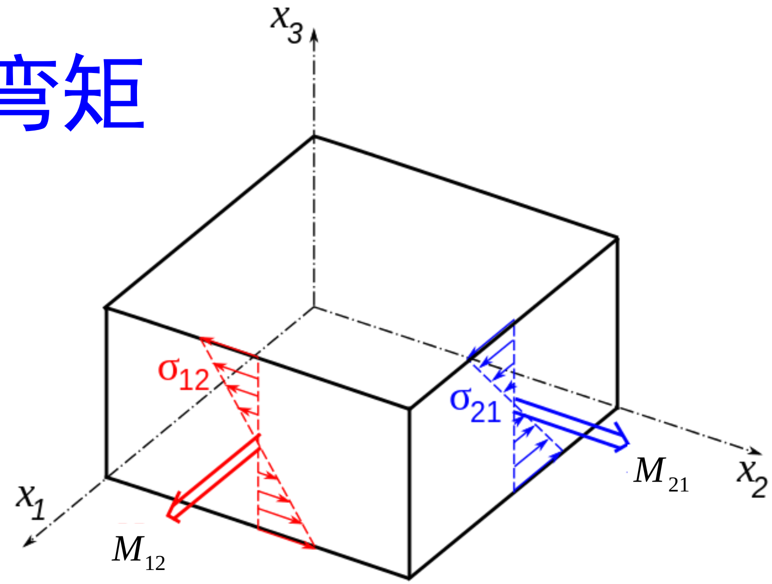
x3
x2
x1
σ11
M11
M22
σ22

弯矩



x3
x2
x1
Q1
σ13
σ23
Q2

剪力



x3
x2
x1
σ12
M12
σ21
M21

扭矩

平衡方程

薄膜力平衡

$$N_{ij,j}=\int_{-h/2}^{h/2}\sigma_{ij,j}dx_3=0$$

$$i,j=1\text{ or }2$$

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{21}}{\partial x_2}=0$$

$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}=0$$

弯曲内力平衡

$$\frac{\partial Q_{13}}{\partial x_1}+\frac{\partial Q_{23}}{\partial x_2}=q$$

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}=Q_{13}$$

$$\frac{\partial M_{21}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}=Q_{23}$$

$$\frac{\partial^2 M_{11}}{\partial x_1{}^2}+2\frac{\partial^2 M_{12}}{\partial x_1\partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2{}^2}=q$$

应力应变关系

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{31} \\ \sigma_{32} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ 2\varepsilon_{31} \\ 2\varepsilon_{32} \end{bmatrix}$$

正交各向异性弹性本构关系

Orthotropic material

应力应变关系

$$\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} dx_3 = h \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ 2\varepsilon_{12}^0 \end{bmatrix}$$

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} x_3 dx_3 = -\frac{h^3}{12} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \varphi_{1,1} \\ \varphi_{2,2} \\ \varphi_{1,2} + \varphi_{2,1} \end{bmatrix}$$

$$\begin{bmatrix} Q_{31} \\ Q_{23} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} C_{44} & 0 \\ 0 & C_{55} \end{bmatrix} \begin{bmatrix} 2\varepsilon_{31} \\ 2\varepsilon_{32} \end{bmatrix} dx_3 = k_s h \begin{bmatrix} C_{44} & 0 \\ 0 & C_{55} \end{bmatrix} \begin{bmatrix} w_{,1} - \varphi_1 \\ w_{,2} - \varphi_2 \end{bmatrix}$$

应力应变关系

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{31} \\ \sigma_{32} \end{bmatrix} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} 2\varepsilon_{31} \\ 2\varepsilon_{32} \end{bmatrix}$$

各向同性弹性本构关系

isotropic material

应力应变关系

$$\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{bmatrix} = \frac{Eh}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ 2\varepsilon_{12}^0 \end{bmatrix} \xrightarrow{\text{中性面}} \begin{bmatrix} \sigma_{11}^0 \\ \sigma_{22}^0 \\ \sigma_{12}^0 \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ 2\varepsilon_{12}^0 \end{bmatrix}$$

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = -\frac{Eh^3}{12\left(1-\nu^2\right)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varphi_{1,1} \\ \varphi_{2,2} \\ \varphi_{1,2}+\varphi_{2,1} \end{bmatrix} = -D_0 \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varphi_{1,1} \\ \varphi_{2,2} \\ \varphi_{1,2}+\varphi_{2,1} \end{bmatrix}$$

$$\begin{bmatrix} Q_{31} \\ Q_{23} \end{bmatrix} = k_s h \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} w_{,1}-\varphi_1 \\ w_{,2}-\varphi_2 \end{bmatrix} = k_s G h \begin{bmatrix} w_{,1}-\varphi_1 \\ w_{,2}-\varphi_2 \end{bmatrix}$$

代入平衡方程

薄膜力平衡 $N_{ij,j} = 0 \Rightarrow \sigma^0_{ij,j} = 0$

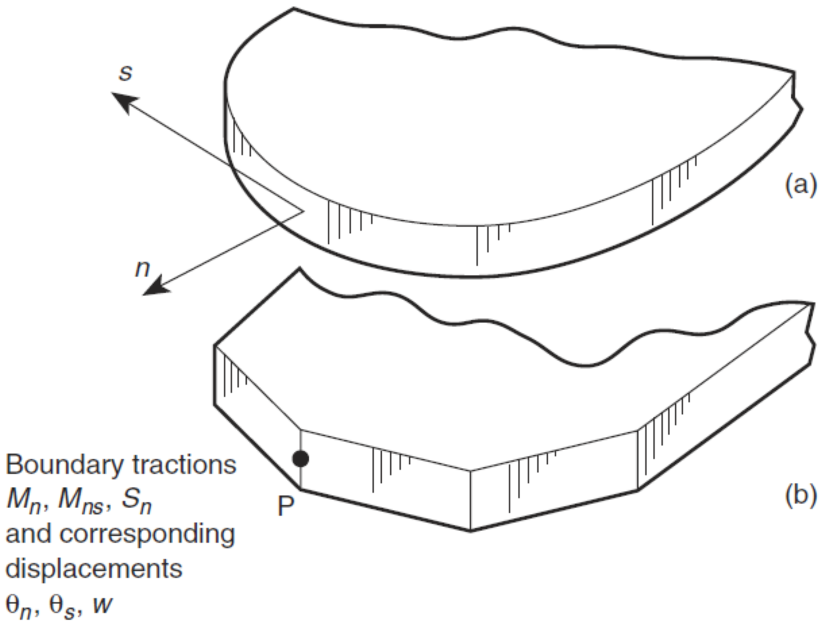
弯曲内力平衡

$$\frac{\partial Q_{13}}{\partial x_1} + \frac{\partial Q_{23}}{\partial x_2} = q$$

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} = Q_{13} \qquad \frac{\partial M_{21}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} = Q_{23}$$

$$\bar{M} := D_0\left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2}\right) \qquad \begin{cases} \nabla^2 \bar{M} = -q \\ k_s G h\left(\nabla^2 w - \dfrac{\bar{M}}{D_0}\right) = -q \end{cases}$$

边界条件



on S

$$\mathbf{n},\mathbf{s} \rightarrow (n,s)$$

(1) $w\big|_S = \bar{w}$

(2) $\varphi_n\big|_S = \bar{\varphi}_n \qquad \varphi_s\big|_S = \bar{\varphi}_s$

(3) $V\big|_S = \bar{V}$

(4) $M_n\big|_S = \bar{M}_n \qquad M_s\big|_S = \bar{M}_s$

(1),(2) (1),(4) (3),(4)

厚板单元

Thick plate element

弱形式

$$\int_{\Omega}\delta\varphi\left(\nabla^{2}\bar{M}+q\right)\mathrm{d}\Omega+\int_{\Omega}\delta w\left[k_{s}Gh\left(\nabla^{2}w-\frac{\bar{M}}{D_{0}}\right)+q\right]\mathrm{d}\Omega=0$$

$\delta w=0$ $\delta\varphi=0$ 相应位移边界

势能泛函

$$\Pi=\int_{\Omega}\left(\frac{1}{2}\boldsymbol{\kappa}^{\mathrm{T}}\mathbf{D}\boldsymbol{\kappa}\right)\mathrm{d}\Omega-\int_{\Omega}qw\mathrm{d}\Omega-\int_{\Gamma_V}\bar{V}w\,dS+\int_{\Gamma_M}\bar{\mathbf{M}}\boldsymbol{\varphi}\,dS+\int_{\boldsymbol{\Omega}}\frac{k_s}{2}\mathbf{Q}^{\mathrm{T}}\mathbf{C}\mathrm{d}\boldsymbol{\Omega}$$

$$\boldsymbol{\kappa}=-\begin{bmatrix}\varphi_{1,1}\\ \varphi_{2,2}\\ \varphi_{1,2}+\varphi_{2,1}\end{bmatrix}=\mathbf{L}\boldsymbol{\varphi}\quad \mathbf{Q}=\begin{bmatrix}Q_{31}\\ Q_{32}\end{bmatrix}\qquad \mathbf{C}=\begin{bmatrix}w_{,1}-\varphi_1\\ w_{,2}-\varphi_2\end{bmatrix}$$

$$\int_{\Omega}\frac{k_s}{2}\mathbf{Q}^{\mathrm{T}}\mathbf{C}\mathrm{d}\boldsymbol{\Omega}=\int_{\Omega}\frac{\alpha_s}{2}\left(w_{,1}-\varphi_1\right)^2 d\Omega+\int_{\Omega}\frac{\alpha_s}{2}\left(w_{,2}-\varphi_2\right)^2 d\Omega$$

$$\alpha_s=k_s G h$$

势能泛函

$$\delta\Pi=\int_{\Omega}\mathbf{D}\delta\boldsymbol{\kappa}\mathrm{d}\Omega-\int_{\Omega}q\delta w\mathrm{d}\Omega-\int_{\Gamma_V}\bar{V}\delta w dS+\int_{\Gamma_M}\bar{\mathbf{M}}\delta\boldsymbol{\varphi}dS$$

$$+\int_{\Omega}\alpha_s\delta\left(w_{,1}-\varphi_1\right)d\Omega+\int_{\Omega}\alpha_s\delta\left(w_{,2}-\varphi_2\right)d\Omega=0$$

$$\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}=\mathbf{N}_{\varphi}\begin{bmatrix}\tilde{\varphi}_1\\ \tilde{\varphi}_2\end{bmatrix}$$

$$w=\mathbf{N}_w\tilde{w}$$

分别插值

⇒

$$\mathbf{Ka}=\mathbf{P}$$

$$\mathbf{K}=\mathop{\mathsf{A}}_{e=1}^{n_{el}}\mathbf{K}^e,\mathbf{P}=\mathop{\mathsf{A}}_{e=1}^{n_{el}}\mathbf{P}^e$$

单元刚度与等效荷载

$$\mathbf{K}^e = \mathbf{K}_b^e + \mathbf{K}_s^e$$

flexture　　shear

$$\mathbf{K}_b^e = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{\varphi\varphi}^b \end{bmatrix} \qquad \mathbf{K}_s^e = \begin{bmatrix} \mathbf{K}_{ww}^s & \mathbf{K}_{w\varphi}^s \\ \mathbf{K}_{\varphi w}^s & \mathbf{K}_{\varphi\varphi}^s \end{bmatrix}$$

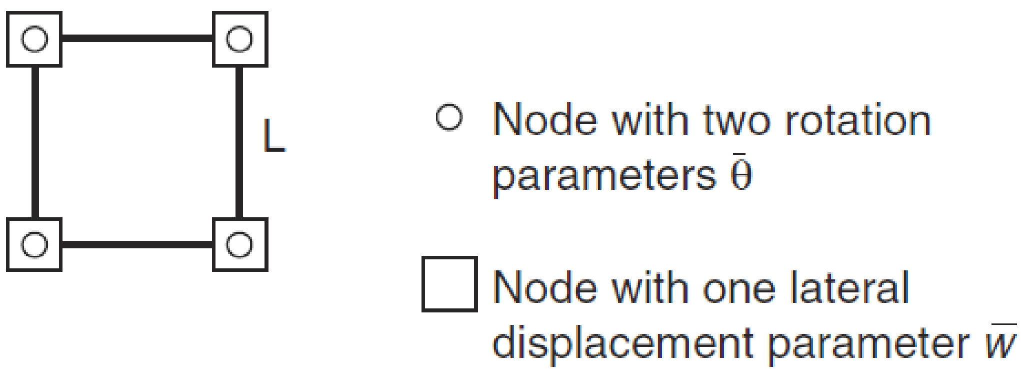
$$\mathbf{K}_{\varphi\varphi}^b = \int_{\Omega^e} \left(\mathbf{L}\mathbf{N}_\varphi\right)^{\mathrm{T}} \mathbf{D} \left(\mathbf{L}\mathbf{N}_\varphi\right) \mathrm{d}\Omega \qquad \mathbf{K}_{ww}^s = \int_{\Omega^e} \left(\nabla\mathbf{N}_w\right)^{\mathrm{T}} \alpha_s \left(\nabla\mathbf{N}_w\right) \mathrm{d}\Omega$$

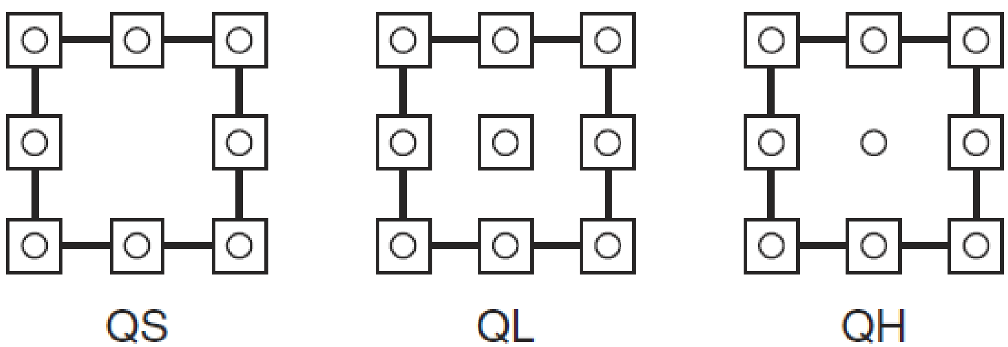
$$\mathbf{K}_{\varphi\varphi}^s = \int_{\Omega^e} \mathbf{N}_\varphi^{\mathrm{T}} \alpha_s \mathbf{N}_\varphi \mathrm{d}\Omega \qquad \mathbf{K}_{\varphi s}^s = \int_{\Omega^e} \mathbf{N}_\varphi^{\mathrm{T}} \alpha_s \left(\nabla\mathbf{N}_w\right) \mathrm{d}\Omega = \left(\mathbf{K}_{s\varphi}^s\right)^{\mathrm{T}}$$

$$\mathbf{P}_w^e = \int_{\Omega^e} \mathbf{N}_w^{\mathrm{T}} q \mathrm{d}\Omega + \int_{\Gamma_V^e} \mathbf{N}_w^{\mathrm{T}} \bar{V} dS \qquad \mathbf{P}_\varphi^e = -\int_{\Gamma_{\mathrm{M}}^e} \mathbf{N}_\varphi^{\mathrm{T}} \bar{\mathbf{M}} dS$$

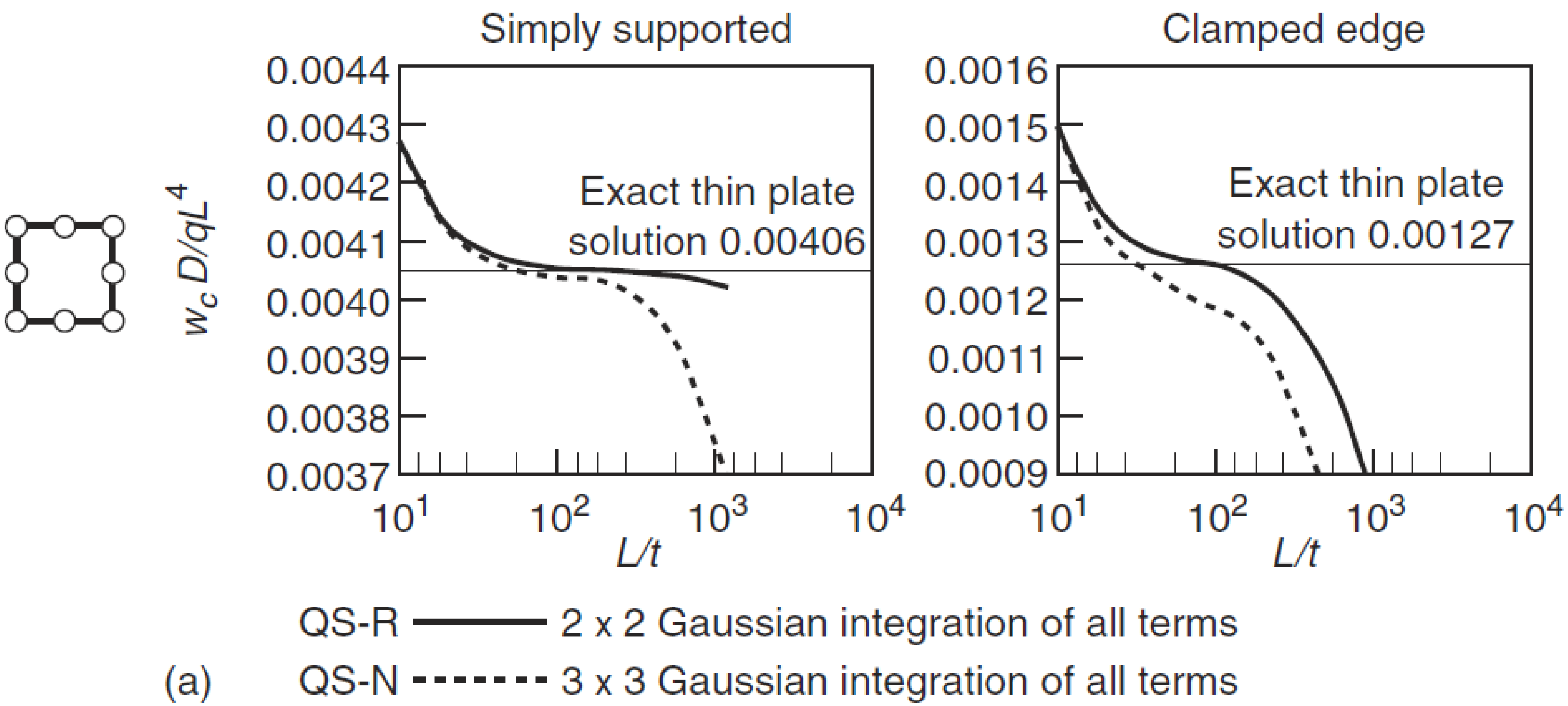
古典厚板单元

采用平面单元插值函数分别对挠度和两个转角进行插值。





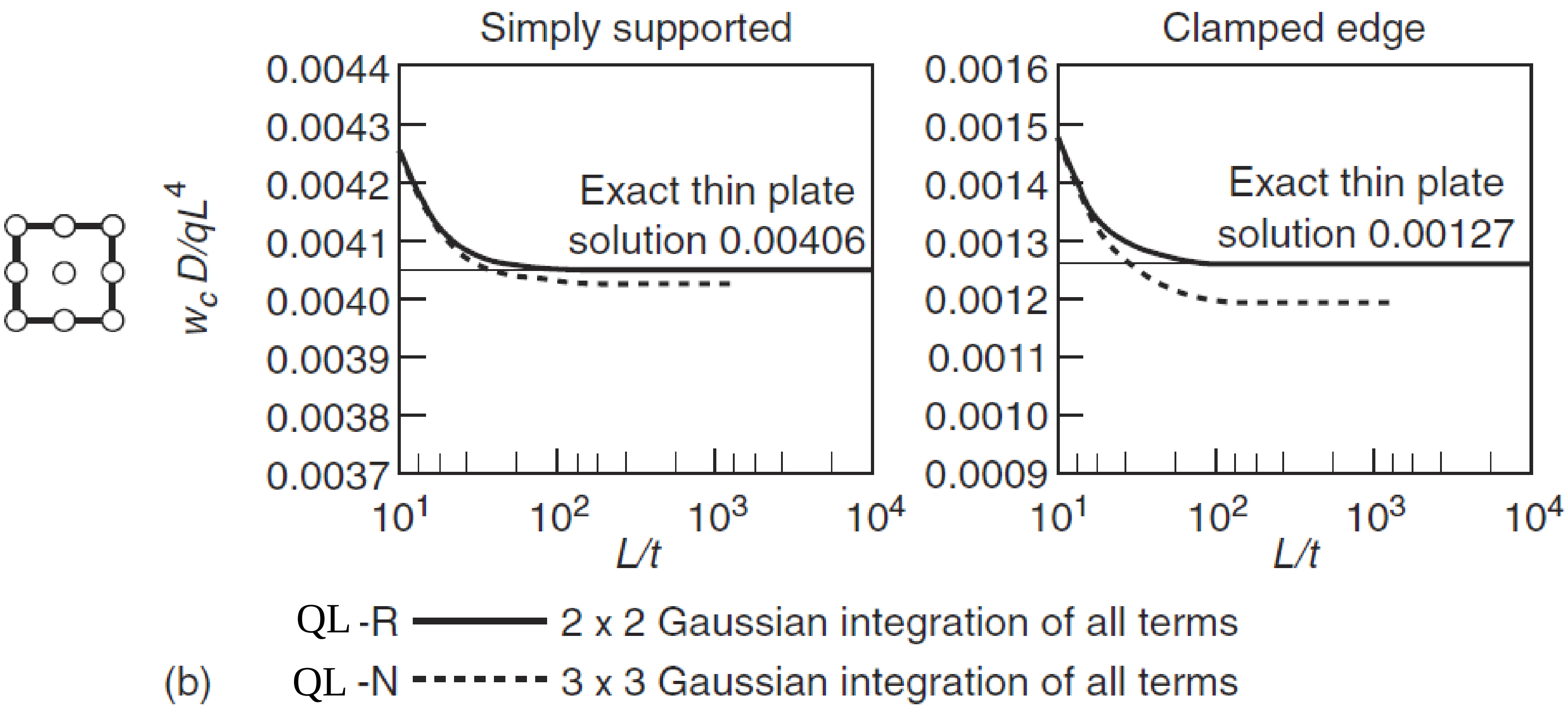
古典厚板单元



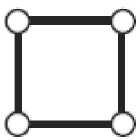
Simply supported
Clamped edge
$w_c D/qL^4$
0.0044
0.0043
0.0042
0.0041
0.0040
0.0039
0.0038
0.0037
Exact thin plate solution 0.00406
0.0016
0.0015
0.0014
0.0013
0.0012
0.0011
0.0010
0.0009
Exact thin plate solution 0.00127
10^1
10^2
10^3
10^4
L/t
QS-R 2 x 2 Gaussian integration of all terms
QS-N 3 x 3 Gaussian integration of all terms

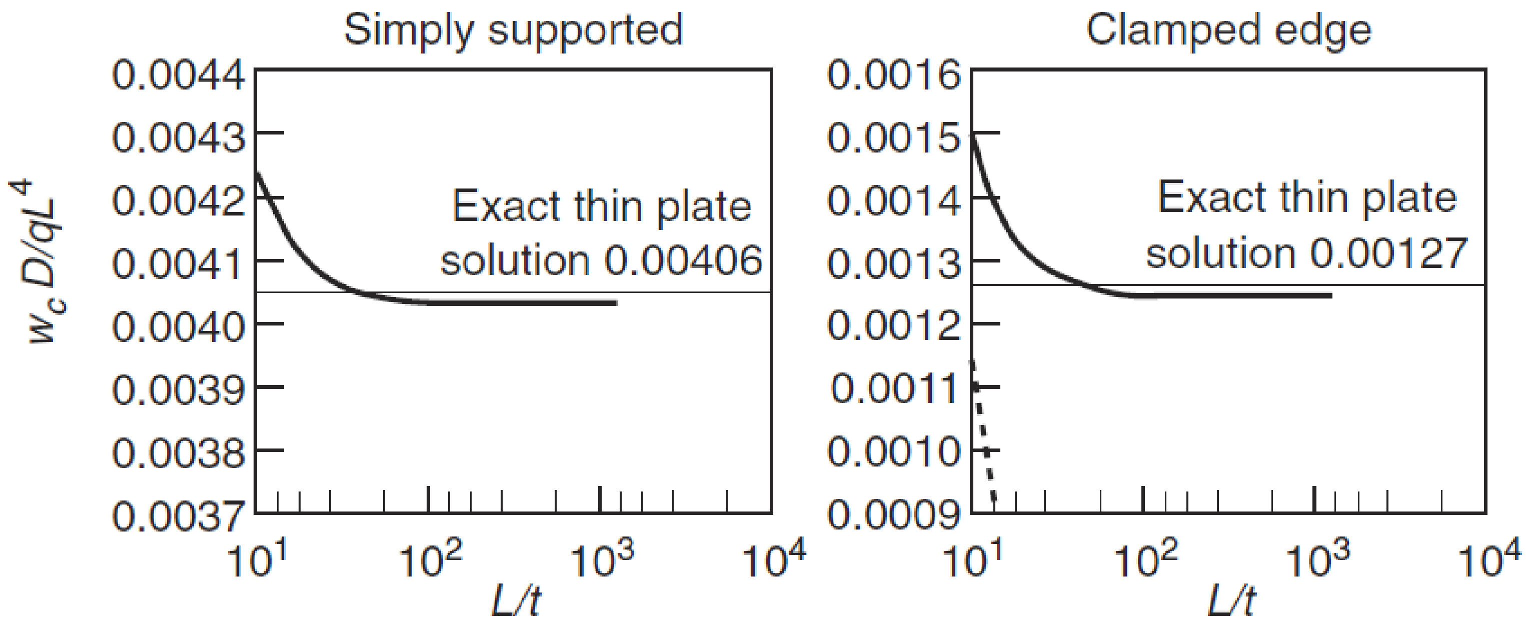
(a)

古典厚板单元


Simply supported
Clamped edge
$w_c D/qL^4$
0.0044
0.0043
0.0042
0.0041
0.0040
0.0039
0.0038
0.0037
0.0016
0.0015
0.0014
0.0013
0.0012
0.0011
0.0010
0.0009
Exact thin plate solution 0.00406
Exact thin plate solution 0.00127
10^1 10^2 10^3 10^4
L/t
QL -R 2 x 2 Gaussian integration of all terms
QL -N 3 x 3 Gaussian integration of all terms
(b)

古典厚板单元





Selective reduced integration

L-R ——— 2 x 2 flexure integration – 1 x 1 shear integration

L-N - - - - - - 2 x 2 integration of all terms – this gives poor results, and diverges rapidly as *L/t* increases

古典厚板单元的使用应该非常小心，需要针对实际情况谨慎验证锁闭和零能模式等问题。

离散配点板单元

$$\Pi=\int_{\Omega}\frac{1}{2}(\mathbf{L\varphi})^{\mathrm{T}}\mathbf{D}(\mathbf{L\varphi})\mathrm{d}\Omega-\int_{\Omega}qw\mathrm{d}\Omega-\int_{\Gamma_V}\bar{V}wdS+\int_{\Gamma_M}\bar{\mathbf{M}}\boldsymbol{\varphi}dS\rightarrow\min$$

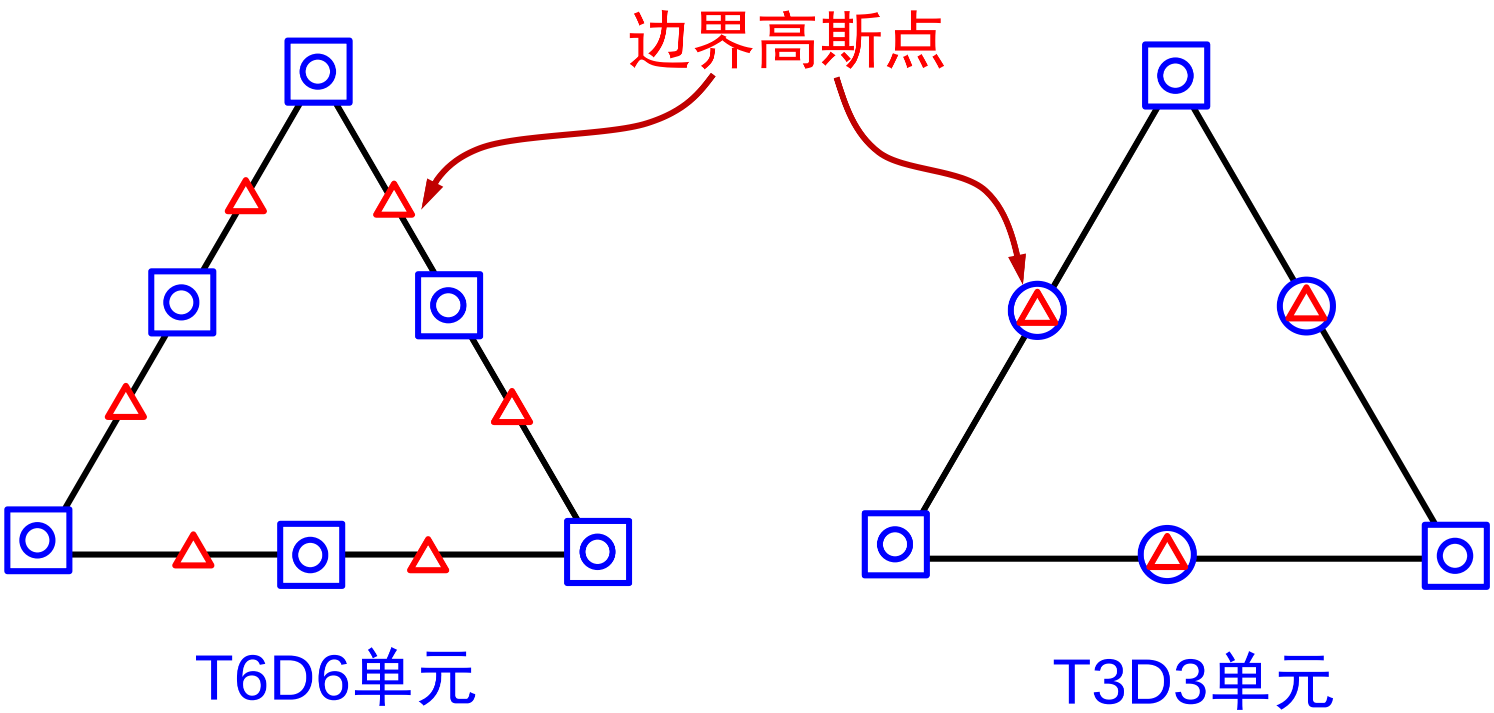
Subject to $\boldsymbol{\varphi}-\nabla w=0$

$$\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}=\mathbf{N}_{\varphi}\begin{bmatrix}\tilde{\varphi}_1\\ \tilde{\varphi}_2\end{bmatrix}\qquad w=\mathbf{N}_w\tilde{w}$$

位移、转角分别插值

约束条件在离散点上实现（discrete Kirchhoff theory, DKT）。实现方法可采用配点法、子域法等，可以数值实现、也可以基于自由度凝聚精确实现。

离散配点板单元



□ Node with one lateral displacement

△ Collocation point for tangent rotation

○ Node with two rotations

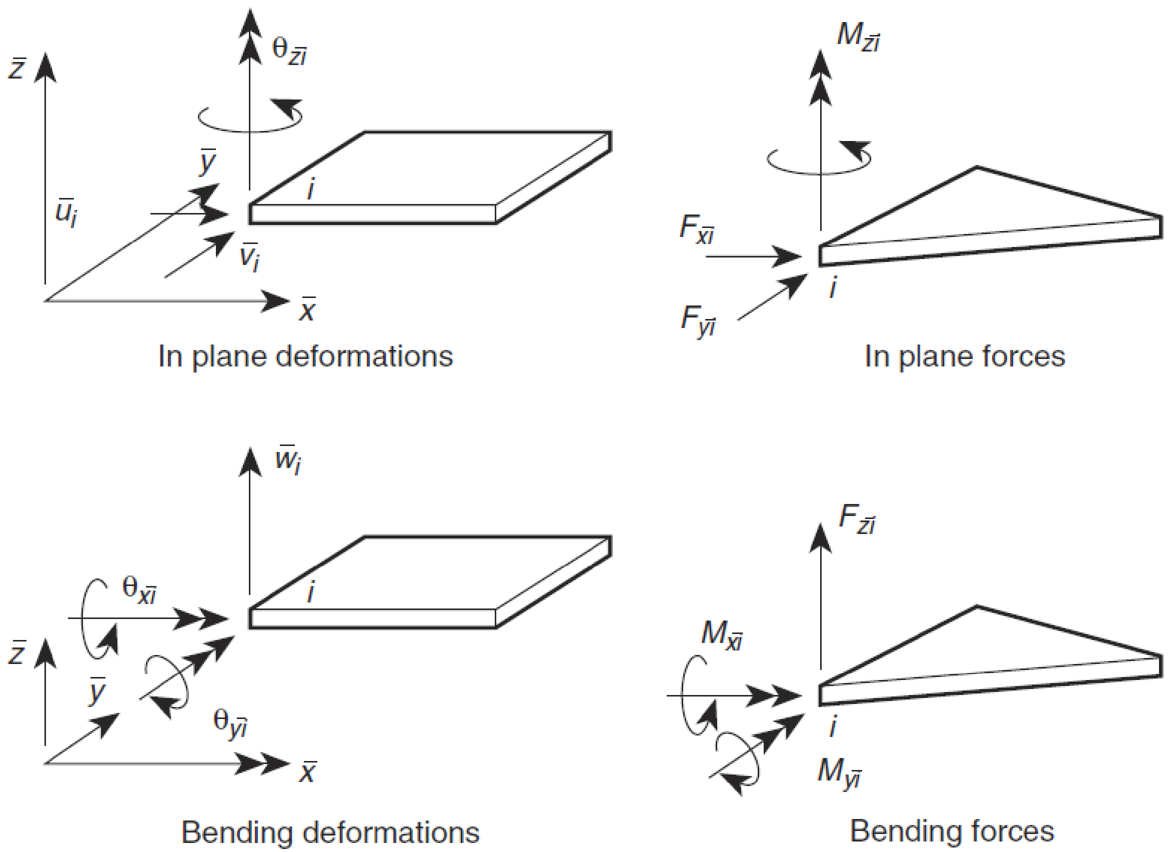
$$\left.\frac{\partial w}{\partial s}\right|_k = \varphi_{sk}$$

应力杂交板单元

根据个人兴趣自学！

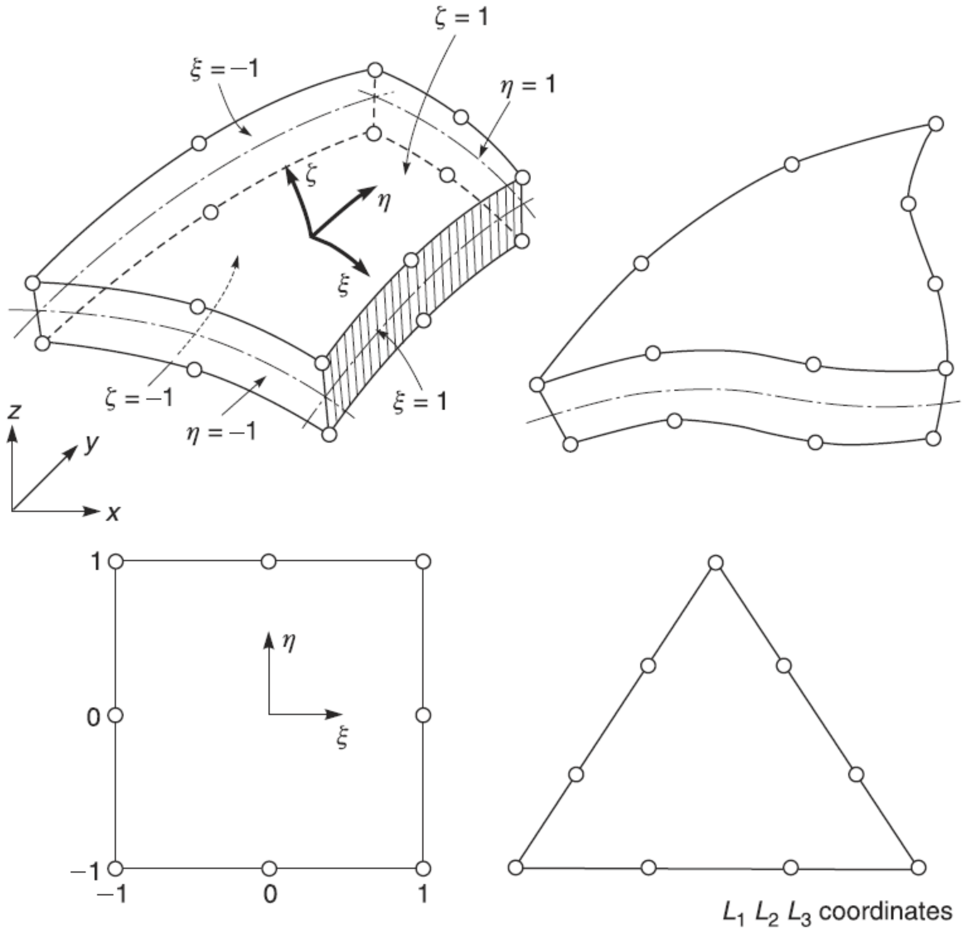
平板壳单元

平面内转动（drilling）常不考虑！



A flat element subject to ‘in-plane’ and ‘bending’ actions.

曲壳单元



Curved thick shell elements of various types.

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今天就到这里，

明天的事儿明天再说！

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